PROOF OF FORMULA 4.227.16

$$\int_0^{\pi/4} \ln^2(\cot x - \tan x)\, dx = \frac{\pi^3}{16} + \frac{\pi}{4}\ln^2 2$$

Start with

$$\cot t - \tan t = \frac{1}{2}\tan 2x$$

and let $u = 2x$ to obtain

$$\int_0^{\pi/4} \ln^2(\cot x - \tan x)\, dx = \frac{1}{2}\int_0^{\pi/2} \ln^2\left(\tfrac{1}{2}\tan u\right)\, du.$$

Expand to obtain

$$\int_0^{\pi/4} \ln^2(\cot x - \tan x)\, dx = \frac{1}{2}\int_0^{\pi/2} \ln^2 \tan u\, du - \ln 2\int_0^{\pi/2} \ln\tan u\, du + \frac{\pi}{2}\ln^2 2.$$

The first integral vanishes according to entry 4.227.3 with $a = 1$. Split the second integral as

$$\int_0^{\pi/2} \ln^2 \tan u\, du = \int_0^{\pi/4} \ln^2 \tan u\, du + \int_{\pi/4}^{\pi/2} \ln^2 \tan u\, du = 2\int_0^{\pi/4} \ln^2 \tan u\, du,$$

after making the change of variables $u \mapsto \pi/2 - u$ in the last integral. Now use entry 4.227.7

$$\int_0^{\pi/4} \ln^2 \tan u\, du = \frac{\pi^3}{16}$$

to obtain the result.