

PROOF OF FORMULA 4.231.10

$$\int_0^\infty \frac{\ln x \, dx}{a^2 - b^2 x^2} = -\frac{\pi^2}{4ab}$$

This has to be interpreted as a Cauchy principal value

$$\int_0^\infty \frac{\ln x \, dx}{a^2 - b^2 x^2} = \lim_{\epsilon \rightarrow 0} \left[\int_0^{a/b-\epsilon} \frac{\ln x \, dx}{a^2 - b^2 x^2} + \int_{a/b+\epsilon}^\infty \frac{\ln x \, dx}{a^2 - b^2 x^2} \right].$$

The change of variable $x = at/b$ gives

$$\int_0^\infty \frac{\ln x \, dx}{a^2 - b^2 x^2} = \frac{1}{ab} \lim_{\epsilon \rightarrow 0} \left[\int_0^{1-b\epsilon/a} \frac{\ln(bt/a) \, dt}{1-t^2} + \int_{1+b\epsilon/a}^\infty \frac{\ln(bt/a) \, dt}{1-t^2} \right].$$

Let $s = 1/t$ in the second integral to get

$$\int_0^\infty \frac{\ln x \, dx}{a^2 - b^2 x^2} = \frac{1}{ab} \lim_{\epsilon \rightarrow 0} \left[\int_0^{(a-b\epsilon)/a} \frac{\ln(bt/a) \, dt}{1-t^2} + \int_0^{a/(a+b\epsilon)} \frac{\ln(at/b) \, dt}{1-t^2} \right].$$

Each integral is evaluated by the method of partial fractions to obtain

$$\begin{aligned} \int_0^\infty \frac{\ln x \, dx}{a^2 - b^2 x^2} &= \frac{\ln(b/a)}{ab} \lim_{\epsilon \rightarrow 0} \left[\int_0^{(a-b\epsilon)/a} \frac{dt}{1-t^2} - \int_0^{a/(a+b\epsilon)} \frac{dt}{1-t^2} \right] + \\ &+ \frac{1}{ab} \lim_{\epsilon \rightarrow 0} \left[\int_0^{(a-b\epsilon)/a} \frac{\ln t \, dt}{1-t^2} + \int_0^{a/(a+b\epsilon)} \frac{\ln t \, dt}{1-t^2} \right]. \end{aligned}$$

The integrands containing $\ln t$ converge to

$$\frac{2}{ab} \int_0^1 \frac{\ln t \, dt}{1-t^2} = -\frac{\pi^2}{4ab},$$

using entry 4.231.13. The other two terms integrate to

$$\frac{1}{2} \ln(2a - b\epsilon) - \frac{1}{2} \ln(2a + b\epsilon)$$

that vanishes as $\epsilon \rightarrow 0$.