PROOF OF FORMULA 4.231.12

$$\int_0^1 \frac{\ln x \, dx}{1+x^2} = -\int_1^\infty \frac{\ln x \, dx}{1+x^2} = -G$$

The constant G in the answer is *Catalan constant* defined by the series

$$G = \sum_{j=0}^\infty \frac{(-1)^j}{(2j+1)^2}.$$

The equality of the two integrals comes from $x \mapsto 1/x$. To evaluate the integral, expand the integrand and integrate to obtain

$$\int_0^1 \frac{\ln x \, dx}{1+x^2} = \sum_{j=0}^\infty (-1)^j \int_0^1 x^{2j} \ln x \, dx.$$

The change of variables $t = \ln x$ produces

$$\int_0^1 x^{2j} \ln x \, dx = -\int_0^\infty t e^{-(2j+1)t} \, dt.$$

Now let $s = (2j+1)t$ to obtain

$$\int_0^\infty t e^{-(2j+1)t} \, dt = \frac{1}{(2j+1)^2} \int_0^\infty s e^{-s} \, ds,$$

and this last integral is $\Gamma(2) = 1$. Replace to obtain the result.