PROOF OF FORMULA 4.231.6

$$\int_0^1 \frac{\ln x}{(1+x)^2}\, dx = -\ln 2$$

Differentiating the geometric series

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k,$$

gives

$$\frac{1}{(1+x)^2} = \sum_{k=0}^{\infty} (-1)^k (k+1) x^k.$$

Integrating from 0 to 1 yields

$$\int_0^1 \frac{\ln x}{(1+x)^2}\, dx = \sum_{k=0}^{\infty} (-1)^k (k+1) J_k,$$

where

$$J_k = \int_0^1 x^k \ln x \, dx.$$

The change of variables $x = e^{-t}$ and $s = (1+k)t$ produce

$$J_k = \int_0^1 x^k \ln x \, dx = \int_0^{\infty} t e^{-(k+1)t}\, dt = \frac{1}{(1+k)^2} \int_0^{\infty} s e^{-s}\, ds.$$

This last integral is $\Gamma(2) = 1$. Therefore

$$\int_0^1 \frac{\ln x}{(1+x)^2}\, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = -\ln 2.$$