

**PROOF OF FORMULA 4.231.7**

$$\int_0^\infty \frac{\ln x dx}{(a^2 + b^2 x^2)^n} = \frac{\Gamma(n - \frac{1}{2}) \sqrt{\pi}}{4(n-1)! a^{2n-1} b} \left[ 2 \ln \left( \frac{a}{2b} \right) - \gamma - \psi \left( n - \frac{1}{2} \right) \right]$$

Let  $t = bx/a$  to obtain

$$\int_0^\infty \frac{\ln x dx}{(a^2 + b^2 x^2)^n} = \frac{\ln(a/b)}{a^{2n-1} b} I_n + \frac{1}{a^{2n-1} b} J_n,$$

where

$$I_n = \int_0^\infty \frac{dt}{(1+t^2)^n} \text{ and } J_n = \int_0^\infty \frac{\ln t dt}{(1+t^2)^n}.$$

The change of variables  $u = t^2$  gives

$$I_n = \frac{1}{2} \int_0^\infty \frac{u^{-1/2} du}{(1+u)^n} = \frac{1}{2} B \left( \frac{1}{2}, n - \frac{1}{2} \right) = \frac{\sqrt{\pi} \Gamma(n - \frac{1}{2})}{2(n-1)!}.$$

The integral  $J_n$  is  $f'(0)$ , where

$$f(r) = \int_0^\infty \frac{t^r dt}{(1+t^2)^n}.$$

The change of variables  $u = t^2$  gives

$$f(r) = \frac{1}{2} \int_0^\infty \frac{u^{r/2-1/2} du}{(1+u)^n} = \frac{\Gamma(r/2 + 1/2) \Gamma(n - r/2 - 1/2)}{2(n-1)!}.$$

Logarithmic differentiation gives

$$f'(0) = \frac{\Gamma(n - \frac{1}{2}) \sqrt{\pi}}{4(n-1)!} \left[ \psi \left( \frac{1}{2} \right) - \psi \left( n - \frac{1}{2} \right) \right].$$

The values

$$\psi(n - \frac{1}{2}) = -\gamma - 2 \ln 2 + 2H_{2n-2} - H_{n-1} \text{ and } \Gamma(m + \frac{1}{2}) = \frac{\sqrt{\pi}}{(2m)!} 2^{2m} m!$$

give the result.