

PROOF OF FORMULA 4.233.1

$$\int_0^1 \frac{\ln x \, dx}{1+x+x^2} = \frac{2}{9} \left(\frac{2\pi^2}{3} - \psi' \left(\frac{1}{3} \right) \right)$$

To evaluate the integral we first compute

$$f(a) = \int_0^1 \frac{t^a \ln t}{1-t} dt.$$

Start with

$$\int_0^1 \frac{t^a dt}{(1-t)^{1-\epsilon}} = B(a+1, \epsilon) = \frac{\Gamma(a+1)\Gamma(\epsilon)}{\Gamma(a+1+\epsilon)}.$$

Differentiate with respect to a to produce

$$\int_0^1 \frac{t^a \ln t \, dt}{(1-t)^{1-\epsilon}} = -\epsilon \Gamma(\epsilon) \frac{\Gamma(a+1)}{\Gamma(a+1+\epsilon)} \left(\frac{\psi(a+1+\epsilon) - \psi(a+1)}{\epsilon} \right).$$

Now let $\epsilon \rightarrow 0$ and use $\epsilon \Gamma(\epsilon) \rightarrow 1$ to get

$$\int_0^1 \frac{t^a \ln t}{1-t} dt = -\psi'(a+1).$$

To evaluate the integral, start with

$$\begin{aligned} \int_0^1 \frac{\ln x \, dx}{1+x+x^2} &= \int_0^1 \frac{1-x}{1-x^3} \ln x \, dx \\ &= \int_0^1 \frac{\ln x \, dx}{1-x^3} - \int_0^1 \frac{x \ln x \, dx}{1-x^3} \\ &= \frac{1}{9} \int_0^1 \frac{t^{-2/3} \ln t \, dt}{1-t} - \frac{1}{9} \int_0^1 \frac{t^{-1/3} \ln t \, dt}{1-t} \\ &= \frac{1}{9} \left(\psi' \left(\frac{2}{3} \right) - \psi' \left(\frac{1}{3} \right) \right). \end{aligned}$$

Now use the identity $\psi(1-x) = \psi(x) + \pi \cot \pi x$ and differentiate to get

$$\psi'(1-x) = -\psi'(x) + \frac{\pi^2}{\sin^2 \pi x}.$$

Use this formula to eliminate $\psi'(2/3)$ and obtain the result.