PROOF OF FORMULA 4.233.3

$$\int_0^1 \frac{x \ln x \, dx}{1+x+x^2} = -\frac{1}{54} \left[7\pi^2 - 6\psi'(1/3) \right]$$

Start with

$$\int_0^1 \frac{x \ln x \, dx}{1+x+x^2} = \int_0^1 \frac{x \ln x}{1-x^3} \, dx - \int_0^1 \frac{x^2 \ln x}{1-x^3} \, dx.$$

The change of variables $t = x^3$ gives

$$\int_0^1 \frac{x \ln x \, dx}{1+x+x^2} = \frac{1}{9} \int_0^1 \frac{t^{-1/3} \ln t}{1-t} \, dt - \frac{1}{9} \int_0^1 \frac{\ln t}{1-t} \, dt.$$

Now use the evaluation

$$\psi'(a) = -\int_0^1 \frac{t^{a-1} \ln t}{1-t} \, dt$$

to obtain

$$\int_0^1 \frac{x \ln x \, dx}{1+x+x^2} = -\frac{1}{9}\psi'(2/3) + \frac{\pi^2}{54}.$$

The result now follows from $\psi'(2/3) = -\psi'(1/3) + 4\pi^2/3$.