PROOF OF FORMULA 4.233.4

$$\int_0^1 \frac{x \ln x \, dx}{1-x+x^2} = \frac{1}{6}\left[\frac{5}{6}\pi^2 - \psi'\left(\frac{1}{3}\right)\right]$$

Write the integral as

$$\int_0^1 \frac{x \ln x \, dx}{1-x+x^2} = \int_0^1 \frac{x(1+x)}{1+x^3} \ln x \, dx$$

and let $t = x^3$ to obtain

$$\int_0^1 \frac{x \ln x \, dx}{1-x+x^2} = \frac{1}{9}\int_0^1 \frac{t^{-1/3} \ln t}{1+t} \, dt + \frac{1}{9}\int_0^1 \frac{\ln t}{1+t} \, dt.$$

The second integral is $-\pi^2/12$ from entry 4.231.2. The first one comes from the evaluation

$$\int_0^1 \frac{t^a \ln t}{1+t} \, dt = \frac{1}{4}\left[\psi'\left(\frac{a}{2}+1\right) - \psi\left(\frac{a+1}{2}\right)\right],$$

obtained by differentiating the relation

$$\int_0^1 \frac{t^{a-1} \, dt}{1+t} = \frac{1}{2}\left[\psi\left(\frac{a+1}{2}\right) - \psi\left(\frac{a}{2}\right)\right].$$

Therefore

$$\int_0^1 \frac{x \ln x \, dx}{1-x+x^2} = \frac{1}{36}\psi'\left(\frac{5}{6}\right) - \frac{1}{36}\psi'\left(\frac{1}{3}\right) - \frac{\pi^2}{108}.$$

The result can now be reduced to the stated answer by using

$$\begin{aligned} \psi'(1-x) &= -\psi'(x) + \frac{\pi^2}{\sin^2(\pi x)}, \\ \psi'(x+\tfrac{1}{2}) &= 4\psi'(2x) - \psi'(x). \end{aligned}$$