PROOF OF FORMULA 4.234.1

$$\int_1^\infty \frac{\ln x \, dx}{(1+x^2)^2} = \frac{G}{2} - \frac{\pi}{8}$$

The change of variables $t = 1/x$ gives

$$\int_1^\infty \frac{\ln x \, dx}{(1+x^2)^2} = \int_0^1 \frac{\ln t \, dt}{(1+t^2)^2} - \int_0^1 \frac{\ln t \, dt}{1+t^2}.$$

Entry 4.231.12 states that

$$\int_0^1 \frac{\ln t \, dt}{1+t^2} = -G.$$

To evaluate the other integral start with entry 4.231.11

$$\int_0^a \frac{\ln x \, dx}{x^2+a^2} = \frac{\pi \, \ln a}{4a} - \frac{G}{a}$$

and now write $a = \sqrt{b}$ to get

$$\int_0^{\sqrt{b}} \frac{\ln x \, dx}{x^2+b} = \frac{\pi \, \ln b}{8\sqrt{b}} - \frac{G}{\sqrt{b}}.$$

Differentiate with respect to the parameter b to produce

$$\int_0^{\sqrt{b}} \frac{\ln x \, dx}{(x^2+b)^2} = \frac{\ln b}{8b\sqrt{b}} - \frac{G}{2b\sqrt{b}} - \frac{\pi}{8b\sqrt{b}} + \frac{\pi \, \ln b}{16b\sqrt{b}}.$$

Now put $b = 1$ to obtain

$$\int_0^1 \frac{\ln x \, dx}{(1+x^2)^2} = -\frac{G}{2} - \frac{\pi}{8}.$$

The result follows from here.