PROOF OF FORMULA 4.234.4

$$\int_0^\infty \frac{(1-x^2)}{(1+x^2)^2} \ln x \, dx = -\frac{\pi}{2}$$

Define

$$I(t,a) := \int_0^\infty \frac{x^a \, dx}{(1+tx^2)} = \frac{1}{2} t^{-(a+1)/2} \int_0^\infty \frac{u^{(a-1)/2} \, du}{1+u}.$$

The identity

$$\int_0^\infty \frac{s^{x-1} \, dx}{1+s} \, ds = B(x, 1-x) = \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x},$$

gives

$$I(t,a) = \frac{\pi}{2\cos(\pi a/2)} t^{-(a+1)/2}.$$

Differentiate with respect to the parameter t to produce

$$\int_0^\infty \frac{x^{a+2} \, dx}{(1+tx^2)^2} = \frac{\pi}{4} \frac{(a+1)}{\cos\left(\frac{\pi a}{a}\right)} t^{-(a+3)/2}.$$

The value $t = 1$ gives

$$\int_0^\infty \frac{x^{a+2} \, dx}{(1+x^2)^2} = \frac{\pi(a+1)}{4\cos(\pi a/2)}.$$

Differentiate with respect to the parameter a gives

$$\int_0^\infty \frac{x^{a+2} \ln x}{(1+x^2)^2} \, dx = \frac{\pi}{4\cos^2(\pi a/2)} \left(\cos\left(\frac{\pi a}{2}\right) + \frac{\pi}{2}(a+1) \sin\left(\frac{\pi a}{2}\right) \right).$$

The result now follows by putting $a = -2$ and $a = 0$ to obtain

$$\int_0^\infty \frac{\ln x \, dx}{(1+x^2)^2} = -\frac{\pi}{4} \text{ and } \int_0^\infty \frac{x^2 \ln x \, dx}{(1+x^2)^2} = \frac{\pi}{4}.$$