

PROOF OF FORMULA 4.241.10

$$\int_0^1 x \sqrt{1-x^2} \ln x \, dx = \frac{1}{9}(3 \ln 2 - 4)$$

Let $t = x^2$ to obtain

$$\int_0^1 x \sqrt{1-x^2} \ln x \, dx = \frac{1}{4} \int_0^1 \sqrt{1-t} \ln t \, dt.$$

The change of variables $t = e^{-u}$ now gives

$$\int_0^1 \sqrt{1-t} \ln t \, dt = - \int_0^\infty u e^{-u} \sqrt{1-e^{-u}} \, du.$$

This last integral was evaluated in 3.451.1 and its value is $4(4 - 3 \ln 2)/9$. This gives the result.