

PROOF OF FORMULA 4.241.4

$$\int_0^1 x^{2n+1} \sqrt{1-x^2} \ln x \, dx = \frac{(2n)!!}{(2n+3)!!} \frac{\pi}{2} \left(\sum_{k=1}^{2n+1} \frac{(-1)^k}{k} - \frac{1}{2n+3} + \ln 2 \right)$$

The change of variable $t = x^2$ gives

$$\int_0^1 x^{2n+1} \sqrt{1-x^2} \ln x \, dx = \frac{1}{4} \int_0^1 t^n (1-t)^{1/2} \ln t \, dt.$$

In the proof of formula 4.253.1 it was shown that

$$\int_0^1 t^{a-1} (1-t)^{b-1} \ln t \, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} [\psi(a) - \psi(a+b)].$$

Therefore

$$\int_0^1 x^{2n+1} \sqrt{1-x^2} \ln x \, dx = \frac{\Gamma(n+1)\Gamma(\frac{3}{2})}{4\Gamma(n+\frac{5}{2})} [\psi(n+1) - \psi(n+\frac{5}{2})].$$

The relations

$$\begin{aligned} \psi(n + \frac{5}{2}) &= -\gamma + 2 \sum_{k=1}^{n+2} \frac{1}{2k-1} - 2 \ln 2 \\ \psi(n+1) &= -\gamma + \sum_{k=1}^n \frac{1}{k} \end{aligned}$$

give the result.