

**PROOF OF FORMULA 4.241.5**

$$\int_0^1 \sqrt{(1-x^2)^{2n-1}} \ln x \, dx = -\frac{(2n-1)!! \pi}{(2n)!!} \frac{1}{4} \left( 2 \ln 2 + \sum_{k=1}^n \frac{1}{k} \right)$$

The change of variable  $t = x^2$  gives

$$\int_0^1 \sqrt{(1-x^2)^{2n-1}} \ln x \, dx = \frac{1}{4} \int_0^1 t^{-1/2} (1-t)^{n-1/2} \ln t \, dt.$$

In the proof of formula 4.253.1 it was shown that

$$\int_0^1 t^{a-1} (1-t)^{b-1} \ln t \, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} [\psi(a) - \psi(a+b)].$$

Therefore

$$\int_0^1 \sqrt{(1-x^2)^{2n-1}} \ln x \, dx = \frac{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}{4\Gamma(n+1)} [\psi(\frac{1}{2}) - \psi(n+1)].$$

The relations

$$\begin{aligned} \psi\left(\frac{1}{2}\right) &= -\gamma - 2 \ln 2 \\ \psi(n+1) &= -\gamma + \sum_{k=1}^n \frac{1}{k} \end{aligned}$$

give the result.