PROOF OF FORMULA 4.244.3

$$\int_0^1 \frac{x \ln x \, dx}{\sqrt[3]{(1-x^3)^2}} = \frac{\pi}{3\sqrt{3}} \left(\frac{\pi}{3\sqrt{3}} - \ln 3 \right)$$

The change of variable $t = x^3$ gives

$$\int_0^1 \frac{x \ln x \, dx}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{9} \int_0^1 t^{-1/3}(1-t)^{-2/3} \ln t \, dt.$$

In the proof of formula 4.253.1 it was shown that

$$\int_0^1 t^{a-1}(1-t)^{b-1} \ln t \, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \left[\psi(a) - \psi(a+b) \right].$$

Therefore

$$\int_0^1 \frac{x \ln x \, dx}{\sqrt[3]{(1-x^3)^2}} = \frac{2\pi}{9\sqrt{3}} \left(\psi(\tfrac{2}{3}) - \psi(1) \right).$$

The values

$$\psi\left(\frac{2}{3}\right) = -\gamma + \frac{\pi}{2\sqrt{3}} - \frac{3}{2}\ln 3 \text{ and } \psi(1) = -\gamma$$

give the result.