

**PROOF OF FORMULA 4.245.1**

$$\int_0^1 \frac{x^{4n+1} \ln x \, dx}{\sqrt{1-x^4}} = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{8} \left( \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} - \ln 2 \right)$$

The change of variable  $t = x^4$  gives

$$\int_0^1 \frac{x^{4n+1} \ln x \, dx}{\sqrt{1-x^4}} = \frac{1}{16} \int_0^1 t^{n-1/2} (1-t)^{-1/2} \ln t \, dt.$$

In the proof of formula 4.253.1 it was shown that

$$\int_0^1 t^{a-1} (1-t)^{b-1} \ln t \, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} [\psi(a) - \psi(a+b)].$$

Therefore

$$\int_0^1 \frac{x^{4n+1} \ln x \, dx}{\sqrt{1-x^4}} = \frac{\Gamma(n+1/2)\Gamma(1/2)}{16\Gamma(n+1)} [\psi(n+1/2) - \psi(n+1)].$$

The values

$$\begin{aligned} \Gamma(n + \tfrac{1}{2}) &= \frac{\sqrt{\pi}}{2^n} (2n-1)!! \\ \psi(n + \tfrac{1}{2}) &= -\gamma + 2 \left( \sum_{k=1}^n \frac{1}{2k-1} - \ln 2 \right) \\ \psi(n+1) &= -\gamma + \sum_{k=1}^n \frac{1}{k}, \end{aligned}$$

give the result.