

PROOF OF FORMULA 4.246

$$\int_0^1 (1-x^2)^{n-1/2} \ln x \, dx = -\frac{(2n-1)!!}{(2n)!!} \frac{\pi}{4} \left(2 \ln 2 + \sum_{k=1}^n \frac{1}{k} \right)$$

The change of variable $t = x^2$ gives

$$\int_0^1 (1-x^2)^{n-1/2} \ln x \, dx = \frac{1}{4} \int_0^1 t^{-1/2} (1-t)^{n-1/2} \ln t \, dt.$$

In the proof of formula 4.253.1 it was shown that

$$\int_0^1 t^{a-1} (1-t)^{b-1} \ln t \, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} [\psi(a) - \psi(a+b)].$$

Therefore

$$\int_0^1 (1-x^2)^{n-1/2} \ln x \, dx = \frac{\Gamma(n+1/2)\Gamma(1/2)}{4\Gamma(n+1)} [\psi(1/2) - \psi(n+1)].$$

The values

$$\begin{aligned} \Gamma(n + \frac{1}{2}) &= \frac{\sqrt{\pi}}{2^n} (2n-1)!! \\ \psi(\frac{1}{2}) &= -\gamma - 2 \ln 2 \\ \psi(n+1) &= -\gamma + \sum_{k=1}^n \frac{1}{k}, \end{aligned}$$

give the result.