

**PROOF OF FORMULA 4.251.2**

$$\int_0^\infty \frac{x^{\mu-1} \ln x}{a-x} dx = \pi a^{\mu-1} \left( \cot \pi \mu \ln a - \frac{\pi}{\sin^2 \pi \mu} \right)$$

The change of variables  $x = at$  yields

$$\int_0^\infty \frac{x^{\mu-1} \ln x}{a-x} dx = a^{\mu-1} \ln a \int_0^\infty \frac{t^{\mu-1} dt}{1-t} + a^{\mu-1} \int_0^\infty \frac{t^{\mu-1} \ln t}{1-t} dt.$$

Split the first integral at  $t = 1$  and change  $t$  by  $1/t$  in the range  $t \geq 1$ . Then

$$\int_0^\infty \frac{t^{\mu-1} dt}{1-t} = \int_0^1 \frac{t^{\mu-1} - t^{-\mu}}{1-t} dt.$$

This integral has the value  $\pi \cot \pi \mu$  according to entry 3.231.1.

The same splitting in the second integral gives

$$\int_0^\infty \frac{t^{\mu-1} \ln t}{1-t} dt = \int_0^1 \frac{t^{\mu-1} \ln t}{1-t} dt + \int_0^1 \frac{t^{-\mu} \ln t}{1-t} dt.$$

The result now follows from

$$\int_0^\infty \frac{t^{\mu-1} \ln t}{1-t} dt = -\psi'(\mu)$$

and the identity

$$\psi'(\mu) + \psi'(1-\mu) = \frac{\pi^2}{\sin^2 \pi \mu}.$$