PROOF OF FORMULA 4.252.1

$$\int_0^\infty \frac{x^{\mu-1}\,\ln x\,dx}{(x+\beta)(x+\gamma)} = \frac{\pi}{(\gamma-\beta)\sin\pi\mu}\left[\beta^{\mu-1}\ln\beta - \gamma^{\mu-1}\ln\gamma - \pi\cot\pi\mu\left(\beta^{\mu-1}-\gamma^{\mu-1}\right)\right]$$

Start with the partial fraction decomposition

$$\frac{1}{(x+\beta)(x+\gamma)} = \frac{1}{\gamma-\beta}\frac{1}{x+\beta} - \frac{1}{\gamma-\beta}\frac{1}{x+\gamma}.$$

This gives

$$\int_0^\infty \frac{x^{\mu-1}\,\ln x\,dx}{(x+\beta)(x+\gamma)} = \frac{1}{\gamma-\beta}\int_0^\infty \frac{x^{\mu-1}\,\ln x\,dx}{x+\beta} - \frac{1}{\gamma-\beta}\int_0^\infty \frac{x^{\mu-1}\,\ln x\,dx}{x+\gamma}.$$

The result now follows from formula 4.251.1 that states

$$\int_0^\infty \frac{x^{\mu-1}\,\ln x}{x+b} = \frac{\pi b^{\mu-1}}{\sin\pi\mu}\left(\ln b - \pi\cot\pi\mu\right).$$