PROOF OF FORMULA 4.253.4

$$\int_0^\infty \ln x \left(\frac{x}{a^2 + x^2} \right)^p \frac{dx}{x} = \frac{\ln a}{2a^p} B\left(\frac{p}{2}, \frac{p}{2} \right)$$

Let $x = at$ to obtain

$$\int_0^\infty \ln x \left(\frac{x}{a^2 + x^2} \right)^p \frac{dx}{x} = \frac{\ln a}{a^p} \int_0^\infty \frac{t^{p-1}\, dt}{(1+t^2)^p} + \frac{1}{p} \int_0^\infty \frac{\ln t}{t} \left(\frac{t}{1+t^2} \right)^p dt.$$

The first integral evaluates to $\frac{1}{2} B\left(\frac{p}{2}, \frac{p}{2} \right)$ via the change of variables $s = t^2$ and the integral representation for the beta function

$$B(u, v) = \int_0^\infty \frac{s^{u-1}\, ds}{(1+s)^{u+v}}.$$

The second integral vanishes. This can be seen by splitting the integral at $s = 1$ and making the change of variables $s \mapsto 1/s$ in the second integral.