

PROOF OF FORMULA 4.254.2

$$\int_0^\infty \frac{x^{p-1} \ln x}{1-x^q} dx = -\frac{\pi^2}{q^2 \sin^2(\pi p/q)}$$

Let $t = x^q$ to obtain

$$\int_0^\infty \frac{x^{p-1} \ln x}{1-x^q} dx = \frac{1}{q^2} \int_0^\infty \frac{t^{p/q-1} \ln t}{1-t} dt.$$

In the integral over $[1, \infty)$ let $s = 1/t$ to obtain

$$\int_1^\infty \frac{t^{p/q-1} \ln t}{1-t} dt = \int_0^1 \frac{s^{p/q} \ln s}{1-s} ds.$$

It follows that

$$\int_0^\infty \frac{x^{p-1} \ln x}{1-x^q} dx = \frac{1}{q^2} \int_0^1 \frac{\ln t}{1-t} (t^{p/q-1} + t^{-p/q}) dt.$$

The integral representation for the digamma function

$$\psi(a) = -\int_0^1 \left(\frac{1}{\ln t} + \frac{t^{a-1}}{1-t} \right) dt$$

produces

$$\psi'(a) = -\int_0^1 \frac{t^{a-1} \ln t}{1-t} dt.$$

Therefore

$$\begin{aligned} \int_0^\infty \frac{x^{p-1} \ln x}{1-x^q} dx &= \frac{1}{q^2} \int_0^1 \frac{\ln t}{1-t} (t^{p/q-1} + t^{-p/q}) dt \\ &= -\frac{1}{q^2} \left[\psi' \left(\frac{p}{q} \right) + \psi' \left(1 - \frac{p}{q} \right) \right]. \end{aligned}$$

Differentiating the identity

$$\psi(a) - \psi(1-a) = -\pi \cot \pi a,$$

at $a = p/q$, gives the result.