

PROOF OF FORMULA 4.254.6

$$\int_0^1 \frac{x^{q-1} \ln x}{1-x^{2q}} dx = -\frac{\pi^2}{8q^2}$$

The change of variable $t = x^{2q}$ gives

$$\int_0^1 \frac{x^{q-1} \ln x}{1-x^{2q}} dx = \frac{1}{4q^2} \int_0^1 \frac{t^{-1/2} \ln t}{1-t} dt.$$

The identity

$$\int_0^1 \frac{t^{a-1} \ln t}{1-t} dt = -\psi'(a)$$

appears in the proof of entry 4.252.3. Therefore

$$\int_0^1 \frac{x^{q-1} \ln x}{1-x^{2q}} dx = -\frac{1}{4q^2} \psi' \left(\frac{1}{2} \right).$$

The values

$$\Gamma \left(\frac{1}{2} \right) = \sqrt{\pi}, \Gamma' \left(\frac{1}{2} \right) = \sqrt{\pi} \psi \left(\frac{1}{2} \right), \Gamma'' \left(\frac{1}{2} \right) = \frac{1}{2} \pi^2 \sqrt{\pi} + \sqrt{\pi} \psi' \left(\frac{1}{2} \right),$$

to obtain $\psi' \left(\frac{1}{2} \right) = \frac{1}{2} \pi^2$. Replacing gives the result.