PROOF OF FORMULA 4.261.15

$$\int_0^1 \ln^2 x \, \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{(2n-1)!!\,\pi}{2(2n)!!} \left\{ \frac{\pi^2}{12} + \sum_{k=1}^{2n} \frac{(-1)^k}{k^2} + \left[\sum_{k=1}^{2n} \frac{(-1)^k}{k} + \ln 2 \right]^2 \right\}$$

The change of variables $t = x^2$ gives

$$\int_0^1 \ln^2 x \, \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{1}{8} \int_0^1 \ln^2 t \, t^{n-1/2}(1-t)^{-1/2}\,dt.$$

Entry 4.261.17 states that

$$\int_0^1 \ln^2 x \, x^{\mu-1}(1-x)^{\nu-1}\,dx = B(\mu,\nu)\left[(\psi(\mu)-\psi(\mu+\nu))^2 + \psi'(\mu) - \psi'(\mu+\nu)\right]$$

gives

$$\int_0^1 \ln^2 x \, \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \frac{1}{8} B\left(n+\tfrac{1}{2},\tfrac{1}{2}\right)\left[(\psi(n+1/2)-\psi(n+1))^2 + \psi'(n+1/2) - \psi'(n+1)\right].$$

Now use

$$B\left(n+\frac{1}{2},\frac{1}{2}\right) = \frac{\pi\,(2n)!}{2^{2n}\,(n!)^2}$$

and

$$\begin{aligned}
\psi(n+1) &= -\gamma + \sum_{k=1}^{n} \frac{1}{k} \\
\psi(n+1/2) &= -\gamma + 2\left[\sum_{k=1}^{n} \frac{1}{2k-1} - \ln 2\right] \\
\psi'(n+1) &= \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \\
\psi'(n+1/2) &= \frac{\pi^2}{2} - 4\sum_{k=1}^{n} \frac{1}{(2k-1)^2}
\end{aligned}$$

that appear as entries 8.365.4, 8.366.3, 8.366.11 and 8.366.12, respectively, yield the result.