PROOF OF FORMULA 4.261.16

$$\int_0^1 \ln^2 x \, \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{(2n)!!}{(2n+1)!!} \left\{ -\frac{\pi^2}{12} - \sum_{k=1}^{2n+1} \frac{(-1)^k}{k^2} + \left[\sum_{k=1}^{2n+1} \frac{(-1)^k}{k} + \ln 2 \right]^2 \right\}$$

The change of variables $t = x^2$ gives

$$\int_0^1 \ln^2 x \, \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{1}{8} \int_0^1 t^n (1-t)^{-1/2} \, \ln^2 t \, dt.$$

Entry 4.261.17 states that

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1} \, \ln^2 x \, dx = B(\mu,\nu) \left[(\psi(\mu) - \psi(\mu+\nu))^2 + \psi'(\mu) - \psi'(\mu+\nu) \right]$$

gives

$$\int_0^1 \ln^2 x \, \frac{x^{2n+1}\,dx}{\sqrt{1-x^2}} = \frac{1}{8} B\left(n, \tfrac{1}{2}\right) \left[(\psi(n+1) - \psi(n+3/2))^2 + \psi'(n+1) - \psi'(n+3/2) \right].$$

Now use

$$B(n+1, 1/2) = \frac{2\,(2n)!!}{(2n+1)!!}$$

and

$$\begin{aligned} \psi(n+1) &= -\gamma + \sum_{k=1}^{n} \frac{1}{k} \\ \psi(n+1/2) &= -\gamma + 2\left[\sum_{k=1}^{n} \frac{1}{2k-1} - \ln 2 \right] \\ \psi'(n+1) &= \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \\ \psi'(n+1/2) &= \frac{\pi^2}{2} - 4\sum_{k=1}^{n} \frac{1}{(2k-1)^2} \end{aligned}$$

that appear as entries 8.365.4, 8.366.3, 8.366.11 and 8.366.12, respectively, yield the result.