

PROOF OF FORMULA 4.261.18

$$\int_0^1 \ln^2 x \frac{1-x^{n+1}}{(1-x)^2} dx = 2(n+1)\zeta(3) - 2 \sum_{k=1}^n \frac{n+1-k}{k^3}$$

Differentiate

$$\sum_{k=0}^{\infty} x^k = 1/(1-x)$$

to produce

$$\sum_{k=0}^{\infty} (k+1)x^k = 1/(1-x)^2.$$

Thus,

$$\begin{aligned} \frac{1-x^{n+1}}{(1-x)^2} &= \sum_{k=0}^{\infty} (k+1)x^k - \sum_{k=0}^{\infty} (k+1)x^{k+n+1} \\ &= \sum_{k=0}^{\infty} (k+1)x^k - \sum_{k=n+1}^{\infty} (k-n)x^k \\ &= \sum_{k=0}^n (k+1)x^k + (n+1) \sum_{k=n+1}^{\infty} x^k. \end{aligned}$$

Now use

$$\int_0^1 x^a \ln^2 x dx = \int_0^{\infty} t^2 e^{-(a+1)t} dt = \frac{2}{(a+1)^3},$$

to obtain

$$\begin{aligned} \int_0^1 \ln^2 x \frac{1-x^{n+1}}{(1-x)^2} dx &= \sum_{k=0}^n (k+1) \int_0^1 x^k \ln^2 x dx + (n+1) \sum_{k=n+1}^{\infty} \int_0^1 x^k \ln^2 x dx \\ &= 2 \sum_{k=0}^n \frac{1}{(k+1)^3} + 2 \sum_{k=n+1}^{\infty} \frac{1}{(k+1)^3} \end{aligned}$$

and this reduces to the result.