PROOF OF FORMULA 4.262.8

$$\int_0^1 \ln^3 x \, \frac{1+(-1)^n x^{n+1}}{(1+x)^2} \, dx = -\frac{7(n+1)\pi^4}{120} + 6 \sum_{k=1}^{n} (-1)^{k-1} \frac{n-k+1}{k^4}$$

The identity

$$\frac{1}{(1+x)^2} = \sum_{j=1}^{\infty} (-1)^{j-1} j x^{j-1}$$

comes from differentiating the geometric series for $1/(1+x)$. Therefore

$$\begin{aligned} \frac{1+(-1)^n x^{n+1}}{(1+x)^2} &= \sum_{j=1}^{\infty} (-1)^{j-1} j x^{j-1} - \sum_{j=1}^{\infty} (-1)^{j+n} j x^{j+n} \\ &= \sum_{j=1}^{n+1} (-1)^j (n+1-j) + (n+1) \sum_{j=0}^{\infty} (-1)^j x^j. \end{aligned}$$

It follows that

$$\int_0^1 \ln^3 x \, \frac{1+(-1)^n x^{n+1}}{(1+x)^2} \, dx = \sum_{j=1}^{n} (-1)^j (n+1-j) \int_0^1 x^{j-1} \ln^3 x \, dx + (n+1) \sum_{j=1}^{n} (-1)^j \int_0^1 x^{j-1} \ln^3 x \, dx.$$

Using

$$\int_0^1 x^a \ln^3 x \, dx = -\frac{6}{(a+1)^4},$$

we obtain

$$\int_0^1 \ln^3 x \, \frac{1+(-1)^n x^{n+1}}{(1+x)^2} \, dx = 6(n+1) \sum_{j=1}^{\infty} \frac{(-1)^j}{j^4} - 6 \sum_{j=1}^{n} (-1)^j \frac{n+1-j}{j^4}.$$

This simplifies to the stated answer.