PROOF OF FORMULA 4.267.6

$$\int_0^1 \frac{(1-x)^p}{\ln x}\,dx = \sum_{k=1}^{\infty} (-1)^k \binom{p}{k} \ln(1+k)$$

Expanding the power it follows that

$$\int_0^1 \frac{(1-x)^p}{\ln x}\,dx = \sum_{k=0}^{\infty} (-1)^k \binom{p}{k} \int_0^1 \frac{x^k\,dx}{\ln x}.$$

To evaluate the remaining integral, start from

$$\int_0^1 x^a\,dx = \frac{1}{1+a}$$

and integrate with respect to a from $a=0$ to $a=k$. It follows that

$$\int_0^1 \frac{x^k-1}{\ln x}\,dx = \ln(1+k).$$

Therefore

$$\int_0^1 \frac{(1-x)^p}{\ln x}\,dx = \sum_{k=0}^{\infty} (-1)^k \binom{p}{k} \int_0^1 \frac{x^k-1}{\ln x}\,dx + \sum_{k=0}^{\infty} (-1)^k \binom{p}{k} \int_0^1 \frac{dx}{\ln x}.$$

The result now follows from the vanishing of the second sum: it is $(1-x)^p$ evaluated at $x=1$. The integral of $1/\ln x$ diverges only logarithmically.