PROOF OF FORMULA 4.267.7

$$\int_0^1 \left(\frac{1-x^p}{1-x} - p \right) \frac{dx}{\ln x} = \ln \Gamma(p+1)$$

Entry 3.231.5 states that

$$\int_0^1 \frac{x^{\mu-1} - x^{\nu-1}}{1-x} \, dx = \psi(\nu) - \psi(\mu).$$

Integrate with respect to μ to produce

$$\int_0^1 \frac{x^{b-1} - x^{a-1}}{1-x} \frac{dx}{\ln x} = \ln \left(\frac{\Gamma(a)}{\Gamma(b)} \right).$$

Now write

$$\begin{aligned} \int_0^1 \left(\frac{1-x^p}{1-x} - p \right) \frac{dx}{\ln x} &= \int_0^1 \left(\frac{1-x^p - p(1-x)}{1-x} \right) \frac{dx}{\ln x} \\ &= \ln \frac{\Gamma(p+1)}{\Gamma(1)} - p \ln \frac{\Gamma(2)}{\Gamma(1)} \end{aligned}$$

to obtain the result.