PROOF OF FORMULA 4.269.1

$$\int_0^1 \frac{\sqrt{\ln 1/x}}{1+x^2}\,dx = \frac{\sqrt{\pi}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{3/2}}$$

The change of variables $u = \ln 1/x$ gives

$$\int_0^1 \frac{\sqrt{\ln 1/x}}{1+x^2}\,dx = \int_0^\infty \frac{u^{1/2}\,e^{-u}\,du}{1+e^{-2u}}.$$

Expand the integrand in series to obtain

$$\int_0^\infty \frac{u^{1/2}\,e^{-u}\,du}{1+e^{-2u}} = \sum_{k=0}^{\infty}(-1)^k \int_0^\infty u^{1/2}e^{-(2k+1)u}\,du.$$

The change of variables $t = (2k+1)u$ produces

$$\int_0^\infty \frac{u^{1/2}\,e^{-u}\,du}{1+e^{-2u}} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{3/2}}\int_0^\infty t^{1/2}e^{-t}\,dt.$$

The integral is $\Gamma(3/2) = \sqrt{\pi}/2$. This gives the result.