PROOF OF FORMULA 4.271.5

$$\int_0^\infty \frac{(\ln x)^n \, dx}{1+x^2} = (-1)^n n! \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{n+1}}$$

Expand the integrand as a geometric series to obtain

$$\int_0^\infty \frac{(\ln x)^n \, dx}{1+x^2} = \sum_{k=0}^\infty (-1)^k \int_0^1 x^{2k} (\ln x)^n \, dx.$$

The change of variables $u = -\ln x$ gives

$$\int_0^1 x^{2k} (\ln x)^n \, dx = \int_0^\infty u^n e^{-(2k+1)u} \, du.$$

Then $t = (2k+1)u$ gives the result.