PROOF OF FORMULA 4.272.17

$$\int_0^1 \left(\ln \frac{1}{x} \right)^{p-1} \frac{x^{q-1}\,dx}{1-ax^q} = \frac{\Gamma(p)}{aq^p} \sum_{k=1}^{\infty} \frac{a^k}{k^p}$$

Expand the series to obtain

$$\int_0^1 \left(\ln \frac{1}{x} \right)^{p-1} \frac{x^{q-1}\,dx}{1-ax^q} = \sum_{k=0}^{\infty} a^k \int_0^1 \left(\ln \frac{1}{x} \right)^{p-1} x^{(k+1)q-1}\,dx.$$

The change of variables $t = \ln 1/x$ gives

$$\int_0^1 \left(\ln \frac{1}{x} \right)^{p-1} \frac{x^{q-1}\,dx}{1-ax^q} = \sum_{k=0}^{\infty} a^k \int_0^{\infty} t^{p-1} e^{-t(k+1)q}\,dt.$$

Now let $w = (k+1)qt$ to obtain the result.