PROOF OF FORMULA 4.291.19

$$\int_0^1 \frac{\ln(1+ax)}{1+ax^2}\, dx = \frac{1}{2\sqrt{a}} \tan^{-1} \sqrt{a} \, \ln(1+a)$$

The change of variables $t = \sqrt{a}x$ gives

$$\int_0^1 \frac{\ln(1+ax)}{1+ax^2}\, dx = \frac{1}{\sqrt{a}} \int_0^{\sqrt{a}} \frac{\ln(1+\sqrt{a}t)}{1+t^2}\, dt.$$

The integral appears in entry 4.291.18

$$\int_0^b \frac{\ln(1+bx)\, dx}{1+x^2} = \frac{1}{2} \tan^{-1} b \, \ln(1+b^2)$$

and replace $b = \sqrt{a}$ to obtain the result.