PROOF OF FORMULA 4.291.8

$$\int_0^1 \frac{\ln(1+x)\,dx}{1+x^2} = \frac{\pi}{8}\ln 2$$

Let $x = \tan\varphi$ to obtain

$$\int_0^1 \frac{\ln(1+x)\,dx}{1+x^2} = \int_0^{\pi/4} \ln(1+\tan\varphi)\,d\varphi.$$

Using

$$\sin\varphi + \cos\varphi = \sqrt{2}\sin\left(\varphi + \frac{\pi}{4}\right)$$

the last integral is

$$\begin{aligned}\int_0^{\pi/4} \ln(1+\tan\varphi)\,d\varphi &= \int_0^{\pi/4} \ln(\sin\varphi+\cos\varphi)\,d\varphi - \int_0^{\pi/4} \ln(\cos\varphi)\,d\varphi \\ &= \frac{\pi\ln 2}{8} + \int_{\pi/4}^{\pi/2} \ln\sin\varphi\,d\varphi - \int_0^{\pi/4} \ln\cos\varphi\,d\varphi.\end{aligned}$$

The last two integrals cancel by symmetry.