PROOF OF FORMULA 4.293.2

$$\int_1^\infty x^{\mu-1} \ln(x+1)\, dx = -\frac{1}{\mu} \left[\ln 2 + \beta(-\mu)\right]$$

The change of variables $t = 1/x$ gives

$$\int_1^\infty x^{\mu-1} \ln(x+1)\, dx = \int_0^1 t^{-\mu-1} \ln(1+t)\, dt - \int_0^1 t^{-\mu-1} \ln t\, dt.$$

Entry 4.293.1 shows that the first integral is

$$\int_0^1 t^{-\mu-1} \ln(1+t)\, dt = -\frac{1}{\mu} \left[\ln 2 - \beta(1-\mu)\right].$$

To evaluate the second one, let $t = e^{-s}$ to produce

$$\int_0^1 t^{-\mu-1} \ln t\, dt = -\int_0^\infty s e^{s\mu}\, ds.$$

The parameter $\mu < 0$. Integration by parts shows that this last integral is $-1/\mu^2$. It follows that

$$\int_1^\infty x^{\mu-1} \ln(x+1)\, dx = -\frac{1}{\mu} \left[\ln 2 - \beta(1-\mu) - \frac{1}{\mu}\right].$$

To reduce this expression to the given answer, use the series representation

$$\beta(x) = \sum_{k=0}^\infty \frac{(-1)^k}{x+k},$$

given as entry 8.372.1, to obtain $\beta(x+1) = -\beta(x) + 1/x$. This gives the result.