PROOF OF FORMULA 4.293.3

$$\int_0^\infty x^{\mu-1} \ln(1+x)\, dx = \frac{\pi}{\mu \sin \pi\mu}$$

Entry 4.293.14 with $a = 1$ states that

$$\int_0^\infty \frac{x^{\mu-1} \ln(1+x)}{(1+x)^\nu}\, dx = B(\mu, \nu-\mu) \left[\psi(\nu) - \psi(\nu-\mu)\right].$$

Now we let $\nu \to 0$.

Start with

$$B(\mu, \nu-\mu) = \frac{\Gamma(\mu)\Gamma(\nu-\mu+1)}{(\nu-\mu)\Gamma(\nu+1)}\, \nu$$

and then

$$\nu\psi(\nu) = \frac{\nu^2\Gamma'(\nu)}{\Gamma(\nu+1)}.$$

Differentiate $\Gamma(\nu+1) = \nu\Gamma(\nu)$ to get

$$\Gamma'(\nu+1) = \Gamma(\nu) + \nu\Gamma'(\nu),$$

and conclude that $\nu\psi(\nu) \to -1$ as $\nu \to 0$. Therefore

$$B(\mu, \nu-\mu) \left[\psi(\nu) - \psi(\nu-\mu)\right] \to \frac{\Gamma(\mu)\Gamma(1-\mu)}{\mu}$$

as $\nu \to 0$. This gives the result.