PROOF OF FORMULA 4.333

$$\int_0^\infty e^{-\mu x^2} \ln x \, dx = -\frac{1}{4}(\gamma + \ln 4\mu)\sqrt{\frac{\pi}{\mu}}$$

The change of variables $t = \sqrt{\mu}x$ gives

$$\int_0^\infty e^{-\mu x^2} \ln x \, dx = \frac{1}{\sqrt{\mu}} \int_0^\infty e^{-t^2} \ln t \, dt - \frac{\ln \mu}{2\sqrt{\mu}} \int_0^\infty e^{-t^2} \, dt.$$

The second integral is $\sqrt{\pi}/2$. To compute the first one observe that

$$\int_0^\infty t^a e^{-t^2} \, dt = \frac{1}{2} \int_0^\infty s^{(a-1)/2} e^{-s} \, ds = \frac{1}{2}\Gamma\left(\frac{a+1}{2}\right).$$

Differentiate with respect to the parameter a to obtain

$$\int_0^\infty t^a e^{-t^2} \ln t \, dt = \frac{1}{4}\Gamma\left(\frac{a+1}{2}\right)\psi\left(\frac{a+1}{2}\right).$$

It follows that

$$\int_0^\infty e^{-t^2} \ln t \, dt = \frac{1}{4}\Gamma\left(\frac{1}{2}\right)\psi\left(\frac{1}{2}\right).$$

The value $\psi(\frac{1}{2}) = -\frac{\sqrt{\pi}}{2}(\gamma + 2\ln 2)$, given as entry 8.366 completes the evaluation.