PROOF OF FORMULA 8.258.1

$$\int_0^\infty e^{-bx^2}\,\text{erfc}^2 x\,dx = \frac{1}{\sqrt{\pi b}}\left[2\tan^{-1}\sqrt{b} - \cos^{-1}\left(\frac{1}{1+b}\right)\right]$$

Define

$$f(y) = \int_0^\infty \left(\int_{xy}^\infty e^{-t^2}\,dt\right)^2 e^{-bx^2}\,dx.$$

Then

$$f(0) = \frac{\pi}{4}\int_0^\infty e^{-bx^2}\,dx = \frac{\pi\sqrt{\pi}}{8\sqrt{b}} \text{ and } f(1) = \frac{\pi}{4}\int_0^\infty e^{-bx^2}\,\text{erfc}^2 x\,dx.$$

The evaluation of the current integral comes from $f(1)-f(0) = \int_0^1 f'(y)\,dy$. Observe that

$$\begin{aligned} f'(y) &= -2\int_0^\infty xe^{-bx^2}\int_{xy}^\infty e^{-t^2}\,dt\,e^{-x^2y^2}\,dx \\ &= \int_0^\infty \frac{1}{b+y^2}\frac{d}{dx}\left(e^{-(b+y^2)x^2}\right)\int_{xy}^\infty e^{-t^2}\,dt\,dx. \end{aligned}$$

Integrate by parts to produce

$$f'(y) = \frac{1}{b+y^2}\left[-\frac{\sqrt{\pi}}{2} + y\int_0^\infty e^{-(b+2y^2)x^2}\,dx\right].$$

Now use the value

$$\int_0^\infty e^{-(b+2y^2)x^2}\,dx = \frac{\sqrt{\pi}}{2\sqrt{b+2y^2}}$$

to produce

$$f'(y) = -\frac{\sqrt{\pi}}{2(b+y^2} + \frac{\sqrt{\pi}\,y}{2(y^2+b)\sqrt{2y^2+b}}.$$

Integrate to obtain

$$\int_0^1 f'(y)\,dy = -\frac{\sqrt{2}}{2\sqrt{b}}\tan^{-1}(1/\sqrt{b}) + \frac{\sqrt{\pi}}{8\sqrt{b}}\left[\pi - 4\tan^{-1}\left(\sqrt{\frac{b+2}{b}}\right)\right].$$

Now use $\tan^{-1} z + \tan^{-1}(1/z) = \pi/2$ for $z > 0$ to obtain

$$\int_0^\infty e^{-bx^2}\,\text{erfc}^2 x\,dx = \frac{2}{\sqrt{\pi b}}\tan^{-1}\sqrt{b} - \frac{2}{\sqrt{\pi b}}\tan^{-1}\sqrt{\frac{b+2}{b}}.$$

The result now follows from

$$\cos^{-1} u = 2\tan^{-1}\left(\sqrt{\frac{1-u}{1+u}}\right),$$

with $u = 1/(1+b)$.