SCIENTIA
Series A: *Mathematical Sciences*, Vol. 22 (2012), 129-151
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

The integrals in Gradshteyn and Ryzhik. Part 22: Bessel-K functions

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Abstract. The table of Gradshteyn and Ryzhik contains many integrals that can be evaluated using the modified Bessel function. Some examples are discussed and typos in the table are corrected.

1. Introduction

This paper is part of the collection initiated in [**12**], aiming to evaluate the entries in [**8**] and to provide some context. This table contains a large variety of entries involving the Bessel functions. The goal of the current work is to evaluate some entries in [**8**] where the integrand is an elementary function and the result involves the so-called modified Bessel function of the second kind, denoted by $K_\nu(x)$. Other types of integrals containing Bessel functions will appear in a future publication. This introduction contains a brief description of the Bessel functions. The reader is referred to [**3, 13, 14, 15**] for more information about this class of functions.

The Bessel differential equation

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (x^2-\nu^2)u = 0 \tag{1.1}$$

arises from the solution of Laplace's equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 \tag{1.2}$$

in spherical or cylindrical coordinates. The method of Frobenius shows that, for any $\nu \in \mathbb{R}$, the function

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu+1+k)\,k!}\left(\frac{x}{2}\right)^{\nu+2k} \tag{1.3}$$

2000 *Mathematics Subject Classification.* Primary 33.
Key words and phrases. Integrals, Bessel functions, automatic proofs, method of brackets.
The third author was partially supported by the DDMF project of the Microsoft Research-INRIA Joint Centre. The fourth author wishes to acknowledge the partial support of NSF-DMS 0713836.

solves (1.1). The function $J_\nu(x)$ is called the **Bessel function of the first kind**.

In the case $\nu \notin \mathbb{Z}$, the functions $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent, so they form a basis for the space of solutions to (1.1). If $\nu = n \in \mathbb{Z}$, the relation $J_{-n}(x) = (-1)^n J_n(x)$, shows that a second function is required. This is usually obtained from

$$Y_\nu(x) = \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu}, \tag{1.4}$$

and now $\{J_\nu, Y_\nu\}$ is a basis for all $\nu \in \mathbb{R}$. Naturally, when $\nu \in \mathbb{Z}$, the function $Y_\nu(x)$ has to be interpreted as $\lim_{\mu\to\nu} Y_\mu(x)$. The function $Y_\nu(x)$ is called the **Bessel function of the second kind**.

The **modified Bessel equation**

$$x^2\frac{d^2w}{dx^2} + x\frac{dw}{dx} - (x^2+\nu^2)w = 0 \tag{1.5}$$

is solved in terms of the **modified Bessel functions**

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\nu+1+k)\,k!}\left(\frac{x}{2}\right)^{\nu+2k} \tag{1.6}$$

and

$$K_\nu(x) = \frac{\pi}{2}\,\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\pi\nu}. \tag{1.7}$$

As before, if $\nu \in \mathbb{Z}$, the function K_ν has to be replaced by its limiting value. The function $I_\nu(x)$ is called of first kind and $K_\nu(x)$ of second kind. The integral repesentation

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1} e^{-zt}(1-t^2)^{\nu-1/2}\,dt \tag{1.8}$$

appears as entry **3.387.1**. A proof may be found in [**13**].

This paper contains entries in [**8**] that involve the function $K_\nu(x)$ in the answers. For instance, entry **3.324.1**, which is a special case of (2.11), stating that

$$\int_0^\infty \exp\left(-\frac{b}{4x} - ax\right)dx = \sqrt{\frac{b}{a}}K_1(\sqrt{ab}), \tag{1.9}$$

is an example of the type of problems considered here, but entry **6.512.9**, which is

$$\int_0^\infty K_0(ax)J_1(bx)\,dx = \frac{1}{2b}\ln\left(1+\frac{b^2}{a^2}\right), \tag{1.10}$$

where the Bessel function appears in the integrand, will be described in a future publication.

Most of the entries presented here appear in the literature. The objective of this paper is to present several techniques that are applicable to this and other integral evaluations. Some typos in the table [**8**] have been corrected. The work presented here employs a variety of techniques. The choice of method used in a specific entry has been determined by pedagogical as well as efficiency reasons.

Many integrals that appear in this article have integrands that are members of the class of hyperexponential expressions. Recall that $f(x)$ is called hyperexponential if

$f'(x)/f(x) = r(x)$ is a rational function of x. In other words, $f(x)$ satisfies a first-order linear differential equation with polynomial coefficients, namely $q(x)f'(x)-p(x)f(x) = 0$, if we write $r(x) = p(x)/q(x)$. A multivariate function is hyperexponential, if the above property holds for each single variable. Almkvist and Zeilberger [**1**] developed an algorithm for treating integrals with hyperexponential integrand in an automatic fashion. The idea is based on the paradigm of creative telescoping: assume one wants to evaluate the integral $\int_a^b f(x,y)\,dx$. Then the goal of the algorithm is to find a differential equation for f of the following, very special, form

$$c_m(y)\frac{d^m f}{dy^m} + \cdots + c_1(y)\frac{df}{dy} + c_0(y)f = \frac{d}{dx}\left(q(x,y)f\right), \tag{1.11}$$

where the $c_i(y)$ are polynomials and $q(x,y)$ is a bivariate rational function. If one integrates this equation and applies the fundamental theorem of calculus then one obtains a differential equation for the integral. This equation may be used to find a closed form or to prove a certain identity. In many cases, the right-hand side evaluates to zero, yielding a homogeneous o.d.e., in other cases one may end up with an inhomogeneous one. Care has to be taken that all the integrals that appear do really converge (this may not always be the case). The approach just described will be employed and illustrated in Section 7.2.

The Almkvist-Zeilberger algorithm has later been extended to general holonomic functions by Chyzak [**4**]. In this context, a holonomic function is one which satisfies a linear ordinary differential equation with polynomial coefficients for each of its variables (not necessarily of order 1 as in the hyperexponential case). Implementations in Mathematica of these two algorithms are given in the package `HolonomicFunctions` [**10**].

2. A first integral representation of modified Bessel functions

This section describes the integral representations of the modified Bessel function $K_\nu(z)$. A detailed proof of the first result appears as (9.42) in [**13**], page 235.

THEOREM 2.1. *The function $K_\nu(z)$ admits the integral representation*

$$K_\nu(z) = \frac{z^\nu}{2^{\nu+1}}\int_0^\infty t^{-\nu-1}e^{-t-z^2/4t}\,dt. \tag{2.1}$$

This formula appears as entry **8.432.6** *in* [**8**].

REMARK 2.1. Several other entries of [**8**] are obtained by elementary manipulations of (2.1). For instance, it can be written as

$$\int_0^\infty t^{-\nu-1}\exp\left(-t-\frac{b}{t}\right)\,dt = \frac{2}{b^{\nu/2}}K_\nu(2\sqrt{b}). \tag{2.2}$$

EXAMPLE 2.1. Let $b=1$ in (2.2) and make the change of variables $t=e^x$ to obtain

$$\int_{-\infty}^\infty \exp\left(-\nu x - 2\cosh x\right)\,dx = 2K_\nu(2). \tag{2.3}$$

Splitting the integration over $(-\infty, 0)$ and $(0, \infty)$ gives

$$\int_0^\infty \exp\left(-2\cosh x\right)\cosh \nu x\, dx = K_\nu(2). \tag{2.4}$$

EXAMPLE 2.2. Example 2.1 is the special case $\beta = 2$ of entry **3.547.4**:

$$\int_0^\infty \exp\left(-\beta\cosh x\right)\cosh \nu x\, dx = K_\nu(\beta). \tag{2.5}$$

The table employs γ instead of ν. This entry also follows directly from (2.2). The change of variables $t = \sqrt{b}x$ gives

$$\int_0^\infty x^{-\nu-1}\exp\left(-\sqrt{b}\,(x+1/x)\right)\, dx = 2K_\nu(2\sqrt{b}). \tag{2.6}$$

The change of variables $y = e^t$ gives an integral over the whole real line. Splitting the integration as in Example 2.1 produces the result (2.5).

EXAMPLE 2.3. Entry **3.395.1** is

$$\int_0^\infty \left[(\sqrt{x^2-1}+x)^\nu + (\sqrt{x^2-1}+x)^{-\nu}\right]\frac{e^{-\mu x}}{\sqrt{x^2-1}}\, dx = 2K_\nu(\mu). \tag{2.7}$$

The left-hand side of (2.7) transforms as

$$\begin{aligned}
&\int_1^\infty \left[(\sinh\theta+\cosh\theta)^\nu + (\sinh\theta+\cosh\theta)^{-\nu}\right] e^{-\mu\cosh\theta}\, d\theta \\
&\quad = \int_1^\infty \left[e^{\nu\theta} + e^{-\nu\theta}\right] e^{-\mu\cosh\theta}\, d\theta \\
&\quad = 2\int_1^\infty \cosh(\nu\theta) e^{-\mu\cosh\theta}\, d\theta
\end{aligned}$$

and applying (2.5) yields (2.7).

EXAMPLE 2.4. Entry **3.471.12** is

$$\int_0^\infty x^{\nu-1}\exp\left(-x-\frac{\mu^2}{4x}\right)\, dx = 2\left(\frac{\mu}{2}\right)^\nu K_{-\nu}(\mu) \tag{2.8}$$

and it comes directly from (2.2).

EXAMPLE 2.5. The change of variables $s = 1/t$ yields

$$K_\nu(z) = \frac{z^\nu}{2^{\nu+1}}\int_0^\infty s^{\nu-1}e^{-1/s-z^2s/4}\, ds, \tag{2.9}$$

and followed by $s = w/a$ produces

$$K_\nu(z) = \frac{z^\nu}{2^{\nu+1}a^\nu}\int_0^\infty w^{\nu-1}\exp\left(-\frac{a}{w}-\frac{z^2}{4a}w\right)\, dw. \tag{2.10}$$

Now introduce the parameter b by the relation $4ab = z^2$, to obtain

$$\int_0^\infty w^{\nu-1}\exp\left(-\frac{a}{w}-bw\right)\, dw = 2\left(\frac{a}{b}\right)^{\nu/2} K_\nu(2\sqrt{ab}). \tag{2.11}$$

In particular, if $b=1$, it follows that

$$\int_0^\infty w^{\nu-1}\exp\left(-w-\frac{a}{w}\right)\,dw = 2a^{\nu/2}K_\nu(2\sqrt{a}). \tag{2.12}$$

Formula (2.11) appears as entry **3.471.9** of [**8**]. The special case $\nu=1$ is entry **3.324.1** which served as an illustration in (1.9).

Now replace a by b and ν by $-\nu$ in (2.12) to obtain

$$\int_0^\infty w^{-\nu-1}\exp\left(-w-\frac{b}{w}\right)\,dw = \frac{2}{b^{\nu/2}}K_{-\nu}(2\sqrt{b}). \tag{2.13}$$

PROPOSITION 2.1. *The function K_ν satisfies the symmetry*

$$K_\nu(z) = K_{-\nu}(z). \tag{2.14}$$

PROOF. This symmetry is suggested by the differential equation, as only even powers of ν occur. The actual proof follows directly from (1.7). A second proof is obtained by comparing (2.2) with (2.13). □

EXAMPLE 2.6. Entry **3.337.1** is

$$\int_{-\infty}^\infty \exp\left(-\alpha x-\beta\cosh x\right)\,dx = 2K_\alpha(\beta). \tag{2.15}$$

To establish this identity, make the change of variables $t=\beta e^x/2$ to produce

$$\int_{-\infty}^\infty \exp\left(-\alpha x-\beta\cosh x\right)\,dx = \left(\frac{\beta}{2}\right)^\alpha \int_0^\infty t^{-\alpha-1}\exp\left(-t-\frac{\beta^2}{4t}\right)\,dt.$$

The result (2.15) then follows from (2.2) and Proposition 2.1.

EXAMPLE 2.7. The result of Example 2.6 is now employed to produce a proof of the evaluation

$$\int_0^\infty e^{-2b\sqrt{x^2+1}}\,dx = K_1(2b). \tag{2.16}$$

The reader will find the similar looking integral

$$\int_0^\infty e^{-2b\,(x^2+1)^2}\,dx = 2^{-3/2}e^{-b}K_{1/4}(b) \tag{2.17}$$

in Section 7.

The change of variables $t=\sinh x$ produces

$$\begin{aligned}
\int_0^\infty e^{-2b\sqrt{x^2+1}}\,dx &= \int_0^\infty \cosh x\exp\left(-2b\cosh x\right)\,dx \\
&= \frac{1}{2}\int_0^\infty \left(e^x+e^{-x}\right)\exp\left(-2b\cosh x\right)\,dx \\
&= \frac{1}{2}\int_{-\infty}^\infty \exp\left(-x-2b\cosh x\right)\,dx.
\end{aligned}$$

The result then follows from (2.15).

EXAMPLE 2.8. Entry **3.391** is

$$\int_0^\infty [(\sqrt{x+2\beta}+\sqrt{x})^{2\nu}-(\sqrt{x+2\beta}-\sqrt{x})^{2\nu}]e^{-\mu x}dx = 2^{\nu+1}\frac{\nu}{\mu}e^{\beta\mu}K_\nu(\beta\mu). \tag{2.18}$$

Under the change of variables $x \to 2\beta\sinh^2 x$ the left-hand side becomes

$$\begin{aligned}
&(2\beta)^{\nu+1}\int_0^\infty \sinh 2x[(\cosh x+\sinh x)^{2\nu}-(\cosh x-\sinh x)^{2\nu}]e^{-\beta\mu(\cosh 2x-1)}dx\\
&= (2\beta)^{\nu+1}e^{\beta\mu}\int_0^\infty [e^{2\nu x}-e^{-2\nu x}]e^{-2\beta\mu\cosh 2x}\sinh 2x\, dx\\
&= (2\beta)^{\nu+1}e^{\beta\mu}\int_0^\infty [\cosh(\nu+1)x]-\cosh(\nu-1)x]e^{-\beta\mu\cosh x}dx\\
&= \frac{1}{2}(2\beta)^{\nu+1}e^{\beta\mu}\int_{-\infty}^\infty \{\exp[(\nu+1)x-\beta\mu\cosh x]-\exp[-(\nu-1)x-\beta\mu\cosh x]\}dx\\
&= (2\beta)^{\nu+1}e^{\beta\mu}[K_{\nu-1}(\beta\mu)-K_{\nu+1}(\beta\mu)]
\end{aligned}$$

where in the last step Example 2.6 was used. Finally, by the recursion formula for the modified Bessel functions this reduces, as claimed, to the right-hand side of (2.18).

EXAMPLE 2.9. Entry **3.547.2**, given by

$$\int_0^\infty \exp(-\beta\cosh x)\sinh(\gamma x)\sinh x\, dx = \frac{\gamma}{\beta}K_\gamma(\beta), \tag{2.19}$$

follows by rewriting the integral as

$$\begin{aligned}
&2e^{-\beta}\int_0^\infty \exp(-\beta(\cosh 2x-1))\sinh(2\gamma x)\sinh 2x\, dx\\
&= e^\beta\int_0^\infty \exp(-2\beta\sinh^2 x)\left(e^{2\gamma x}-e^{-2\gamma x}\right)\sinh 2x\, dx\\
&= e^\beta\int_0^\infty \exp(-2\beta\sinh^2 x)\left[(\cosh x+\sinh x)^{2\gamma}-(\cosh x-\sinh x)^{2\gamma}\right]\, d(\sinh^2 x)\\
&= e^\beta\int_0^\infty e^{-2\beta u}\left[(\sqrt{u^2+1}+\sqrt{u})^{2\gamma}-(\sqrt{u^2+1}-\sqrt{u})^{2\gamma}\right]\, du
\end{aligned}$$

and applying (2.18).

EXAMPLE 2.10. Entry **3.478.4** is

$$\int_0^\infty x^{\nu-1}\exp\left(-\beta x^p-\gamma x^{-p}\right)\, dx = \frac{2}{p}\left(\frac{\gamma}{\beta}\right)^{\frac{\nu}{2p}}K_{\nu/p}(2\sqrt{\beta\gamma}). \tag{2.20}$$

To evaluate this entry let $y=\beta x^p$ to obtain

$$\int_0^\infty x^{\nu-1}\exp\left(-\beta x^p-\gamma x^{-p}\right)\, dx = \frac{1}{p\beta^{\nu/p}}\int_0^\infty y^{\nu/p-1}e^{-y-\beta\gamma/y}\, dy. \tag{2.21}$$

The value of this last integral is obtained from (2.1).

3. A second integral representation of modified Bessel functions

The next integral representation of the modified Bessel function appears as Entry **3.387.3** of [**8**] and it can also be found as (9.43) in [**13**], page 236. In order to make this paper as self-contained as possible, a proof is presented here.

THEOREM 3.1. *The modified Bessel function K_ν satisfies*

$$\int_1^\infty (x^2-1)^{\alpha-1/2}\, e^{-\mu x}\, dx = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\mu}\right)^\alpha \Gamma(\alpha+\tfrac{1}{2}) K_\alpha(\mu). \tag{3.1}$$

PROOF. Let C be the contour starting at ∞, running along, and just above, the positive real axis to go into a counterclockwise circle of radius larger than 1 about the origin and then back to ∞ along, and just below, the positive real axis. Then

$$\begin{aligned} \oint_C e^{-zt}(t^2-1)^{\nu-1/2}dt &= \oint_C e^{-zt}t^{2\nu-1}(1-t^{-2})^{\nu-1/2}dt \\ &= \sum_{k=0}^\infty \frac{\Gamma(\frac{1}{2}-\nu+k)}{k!\Gamma(\frac{1}{2}-\nu)} \oint_C t^{2\nu-1-2k}e^{-zt}dt. \end{aligned} \tag{3.2}$$

The last integral in (3.2) is Hankel's integral representation for the gamma function, so

$$\begin{aligned} \oint_C e^{-zt}(t^2-1)^{\nu-1/2}dt &= \frac{2\pi i}{\Gamma(\frac{1}{2}-\nu)} \sum_{k=0}^\infty \frac{\Gamma(\frac{1}{2}-\nu+k)z^{2k-2\nu}}{k!\Gamma(2k-2\nu+1)} \\ &= \frac{2^{\nu+1}\pi i e^{-i\nu\pi}\Gamma(1/2)}{\Gamma(\frac{1}{2}-\nu)} \frac{J_{-\nu}(iz)}{(iz)^\nu} \end{aligned} \tag{3.3}$$

Thus,

$$I_{-\nu} = \frac{\Gamma(\frac{1}{2}-\nu)e^{2\pi\nu i}(z/2)^\nu}{2\pi i\Gamma(1/2)} \oint_C e^{-zt}(t^2-1)^{\nu-1/2}dt. \tag{3.4}$$

Since C encloses ± 1, branch points of the integrand at which it vanishes, we can collapse C to the real axis from -1 to ∞ (the branch cut runs from -1 to 1). We have, integrating over the two segments above ($t-1=(1-t)e^{i\pi}$) and below ($t-1=(1-t)e^{-i\pi}$) the positive real axis,

$$I_{-\nu}(z) = \frac{\Gamma(\frac{1}{2}-\nu)e^{2\pi\nu i}(z/2)^\nu}{2\pi i\Gamma(1/2)} \times \tag{3.5}$$

$$\{(1-e^{-4\pi\nu i})\int_1^\infty e^{-zt}(t^2-1)^{\nu-1/2}dt + i(e^{-\pi\nu i}+e^{-3\pi\nu i})\int_{-1}^1 e^{-zt}(1-t^2)^{\nu-1/2}dt\}.$$

Therefore, from (1.8) and (3.5)

$$\frac{I_{-\nu}(z)-I_\nu(z)}{\sin\pi\nu} = \frac{\Gamma(\frac{1}{2}-\nu)}{\pi\Gamma(\frac{1}{2})}\left(\frac{z}{2}\right)^\nu \int_1^\infty e^{-zt}(t^2-1)^{\nu-1/2}dt. \tag{3.6}$$

Consequently, by (1.7),

$$\int_1^\infty e^{-zt}(t^2-1)^{\nu-1/2}dt = \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})}\left(\frac{2}{z}\right)^\nu K_\nu(z). \tag{3.7}$$

This completes the proof. ∎

Several entries of [**8**] are now obtained by simple manipulations of (3.1).

Example 3.1. The scaled version

$$\int_a^\infty (x^2-a^2)^{\nu-1}e^{-\mu x}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2a}{\mu}\right)^{\nu-\frac{1}{2}}\Gamma(\nu)K_{\nu-\frac{1}{2}}(a\mu), \tag{3.8}$$

appears as entry **3.387.6** in [**8**]. To establish this formula, let $t = ax$ to obtain

$$\int_a^\infty (x^2-a^2)^{\nu-1}e^{-\mu x}\,dx = a^\nu\int_1^\infty (t^2-1)^{\nu-1}e^{-\mu a t}\,dt. \tag{3.9}$$

Now use (3.1) with $\alpha = \nu - \frac{1}{2}$ and μa instead of μ.

Example 3.2. The change of variables $x \to \cosh x$ in (3.1) yields entry **3.547.9**:

$$\int_0^\infty \exp(-\beta\cosh x)\sinh^{2\nu}x\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2}{\beta}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)K_\nu(\beta) \tag{3.10}$$

Example 3.3. Entry **3.479.1**, given by

$$\int_0^\infty \frac{x^{\mu-1}\exp\left(-\beta\sqrt{1+x}\right)}{\sqrt{1+x}}\,dx = \frac{2}{\sqrt{\pi}}\left(\frac{\beta}{2}\right)^{1/2-\nu}\Gamma(\nu)K_{\frac{1}{2}-\nu}(\beta), \tag{3.11}$$

comes from (3.1) by the change of variables $t = \sqrt{1+x}$ and the symmetry of K_ν with respect to the order ν.

Example 3.4. Entry **3.462.25** states that

$$\int_0^\infty \frac{\exp\left(-px^2\right)}{\sqrt{a^2+x^2}}\,dx = \frac{1}{2}\exp\left(\frac{a^2p}{2}\right)K_0\left(\frac{a^2p}{2}\right). \tag{3.12}$$

To evaluate this example, let $x = at$ to produce

$$\int_0^\infty \frac{\exp\left(-px^2\right)}{\sqrt{a^2+x^2}}\,dx = \int_0^\infty \frac{\exp\left(-bt^2\right)}{\sqrt{t^2+1}}\,dt, \tag{3.13}$$

with $b = pa^2$. The change of variables $y = t^2+1$ then gives

$$\int_0^\infty \frac{\exp\left(-bt^2\right)}{\sqrt{t^2+1}}\,dt = \frac{e^b}{2}\int_1^\infty \frac{e^{-by}}{\sqrt{y^2-y}}\,dy. \tag{3.14}$$

Now complete the square to write $y^2 - y = (y-1/2)^2 - 1/4$ and let $y - 1/2 = \omega/2$ to obtain

$$\int_0^\infty \frac{\exp\left(-px^2\right)}{\sqrt{a^2+x^2}}\,dx = \frac{1}{2}e^{b/2}\int_1^\infty (\omega^2-1)^{-1/2}e^{-b\omega/2}\,d\omega. \tag{3.15}$$

This is evaluated by taking $\alpha = 0$ and $\mu = b/2$ in (3.1).

EXAMPLE 3.5. After replacing a by $2a$ in the original formulation in [**8**], entry **3.364.3** is given by

$$\int_0^\infty \frac{e^{-px}\,dx}{\sqrt{x(x+2a)}} = e^{ap}K_0(ap). \tag{3.16}$$

To verify this formula, complete the square and define a new variable of integration by $x + \frac{a}{2} = \frac{1}{2}at$. This yields

$$\int_0^\infty \frac{e^{-px}\,dx}{\sqrt{x(x+2a)}} = e^{ap}\int_1^\infty (t^2-1)^{-1/2}e^{-pat}\,dt. \tag{3.17}$$

The result now follows from Theorem 3.1.

EXAMPLE 3.6. Entry **3.383.8** of [**8**] is

$$\int_0^\infty x^{\nu-1}(x+2a)^{\nu-1}e^{-\mu x}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2a}{\mu}\right)^{\nu-\frac{1}{2}} e^{\mu a}\Gamma(\nu)K_{\frac{1}{2}-\nu}(a\mu), \tag{3.18}$$

where we have replaced the original parameter β in [**8**] by $2a$ to simplify the form of the result. To establish this formula, let $t = x + a$ to obtain

$$\int_0^\infty x^{\nu-1}(x+2a)^{\nu-1}e^{-\mu x}\,dx = e^{\mu a}\int_a^\infty (t^2-a^2)^{\nu-1}e^{-\mu t}\,dt. \tag{3.19}$$

The result again follows from Theorem 3.1.

EXAMPLE 3.7. The special case $a = 1$ and $\nu = n + \frac{1}{2}$ and replacing the parameter μ by p in Example 3.6 gives

$$\int_0^\infty x^{n-1/2}(x+2)^{n-1/2}e^{-px}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{2}{p}\right)^n e^p\Gamma(n+\tfrac{1}{2})K_{-n}(p). \tag{3.20}$$

The result is brought to the form

$$\int_0^\infty x^{n-1/2}(2+x)^{n-1/2}e^{-px}\,dx = \frac{(2n-1)!!}{p^n}e^pK_n(p) \tag{3.21}$$

given in entry **3.372** of [**8**], by using the fact that K is an even function of its order and employing the identity

$$(2n-1)!! = \frac{2^n}{\sqrt{\pi}}\Gamma(n+\tfrac{1}{2}). \tag{3.22}$$

This reduction of the double-factorials appears as entry **8.339.2**.

EXAMPLE 3.8. Entry **3.383.3** is

$$\int_a^\infty x^{\mu-1}(x-a)^{\mu-1}e^{-2bx}\,dx = \frac{1}{\sqrt{\pi}}\left(\frac{a}{2b}\right)^{\mu-\frac{1}{2}} \Gamma(\mu)e^{-ab}K_{\mu-\frac{1}{2}}(ab), \tag{3.23}$$

where we have replaced u by a and β by $2b$ to simplify the answer and avoid confusion between u and μ. To prove this, let $t = x - a$ to convert the requested identity into

$$\int_0^\infty t^{\mu-1}(t+a)^{\mu-1}e^{-2bt}\,dt = \frac{1}{\sqrt{\pi}}\left(\frac{a}{2b}\right)^{\mu-\frac{1}{2}} \Gamma(\mu)e^{ab}K_{\mu-\frac{1}{2}}(ab). \tag{3.24}$$

This comes directly from Example 3.6 and the symmetry of $K_\alpha(z)$ respect to α.

EXAMPLE 3.9. Entry **3.388.2** is

$$\int_0^\infty (2\beta x + x^2)^{\nu-1} e^{-\mu x}\, dx = \frac{1}{\sqrt{\pi}} \left(\frac{2\beta}{\mu} \right)^{\nu - \frac{1}{2}} e^{\beta\mu} \Gamma(\nu) K_{\nu-\frac{1}{2}}(\beta\mu). \tag{3.25}$$

This comes directly from Example 3.6.

EXAMPLE 3.10. Entry **3.471.4** states that

$$I = \int_0^a x^{-2\mu}(a-x)^{\mu-1} e^{-\beta/x}\, dx = \frac{1}{\sqrt{\pi a}} \beta^{1/2-\mu} e^{-\beta/2a} \Gamma(\mu) K_{\mu-1/2}\left(\frac{\beta}{2a} \right) \tag{3.26}$$

where we have replaced u by a to avoid confusion. To evaluate this integral, let $t = a/x - 1$ to produce

$$I = \frac{e^{-\beta/a}}{a^\mu} \int_0^\infty t^{\mu-1}(t+1)^{\mu-1} e^{-\beta t/a}\, dt. \tag{3.27}$$

The formula established in Example 3.6 now gives the result.

EXAMPLE 3.11. The proof of entry **3.471.8**,

$$\int_0^a x^{-2\mu}(a^2-x^2)^{\mu-1} e^{-\beta/x}\, dx = \frac{1}{\sqrt{\pi}} \left(\frac{2}{\beta} \right)^{\mu-1/2} a^{\mu-3/2} \Gamma(\mu) K_{\mu-1/2}\left(\frac{\beta}{a} \right), \tag{3.28}$$

is obtained employing the same change of variables as in Example 3.10.

4. A family with typos

Section **3.462** of [**8**] contains five incorrect entries involving the modified Bessel function. There are some typos in both the form of the integrand as well as the value of the integral.

EXAMPLE 4.1. The first entry analyzed here is **3.462.24**: it appears incorrectly as

$$\int_0^\infty \frac{x^{2n} \exp\left(-a\sqrt{x+b^2}\right)}{\sqrt{x^2+b^2}}\, dx = (2n-1)!! \left(\frac{b}{a} \right)^n K_n(ab), \tag{4.1}$$

with the correct version being

$$\int_0^\infty \frac{x^{2n} \exp\left(-a\sqrt{x^2+b^2}\right)}{\sqrt{x^2+b^2}}\, dx = \frac{2^n}{\sqrt{\pi}} \Gamma\left(n + \tfrac{1}{2}\right) \left(\frac{b}{a} \right)^n K_n(ab). \tag{4.2}$$

The argument of the exponential appears incorrectly as $-a\sqrt{x+b^2}$. The presentation in [**8**] also employs the relation (3.22). This becomes inconvenient for $n = 0$.

To confirm (4.1) make the change of variables $t = \sqrt{x^2+b^2}$ to obtain

$$\int_0^\infty \frac{x^{2n} \exp\left(-a\sqrt{x^2+b^2}\right)}{\sqrt{x^2+b^2}}\, dx = \int_b^\infty (t^2-b^2)^{n-1/2} e^{-at}\, dt. \tag{4.3}$$

The result then follows from (3.8).

EXAMPLE 4.2. Entry **3.462.20** states incorrectly that

$$\int_0^\infty \frac{\exp\left(-a\sqrt{x+b^2}\right)}{\sqrt{x^2+b^2}}\,dx = K_0(ab). \tag{4.4}$$

This should be written as

$$\int_0^\infty \frac{\exp\left(-a\sqrt{x^2+b^2}\right)}{\sqrt{x^2+b^2}}\,dx = K_0(ab), \tag{4.5}$$

and follows from (4.2) with $n = 0$.

EXAMPLE 4.3. Entries **3.462.21**, **3.462.22**, **3.462.23** are the special cases of (4.2) with $n = 1, 2, 3$. Each one of these entries has the term $\sqrt{x+b^2}$ instead of the correct $\sqrt{x^2+b^2}$. Entry **3.462.22** has an additional typo in the answer: it has $K_1(ab)$ instead of $K_2(ab)$.

5. The Mellin transform method

The *Mellin transform* of a locally integrable function $f : (0,\infty) \to \mathbb{C}$ is defined by

$$M[f;s] = \tilde{f}(z) = \int_0^\infty t^{s-1} f(t)dt \tag{5.1}$$

whenever the integral converges. Suppose the integral (5.1) converges in a strip $a < \Re s < b$. A function $f(t)$ may be recovered from its Mellin transform $\tilde{f}(s)$ via the *inversion formula*:

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-s}\tilde{f}(s)ds$$

where $a < c < b$.

EXAMPLE 5.1. The Mellin transform of the exponential function $e^{-\mu x}$ is $\mu^{-s}\Gamma(s)$. By the inversion formula, we have, for $s > 0$,

$$e^{-\mu x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\mu^{-s}\Gamma(s)\,ds. \tag{5.2}$$

LEMMA 5.1. *The Mellin transform of $K_\nu(t)$ evaluates as*

$$\int_0^\infty t^{s-1}K_\nu(t)\,dt = 2^{s-2}\Gamma\left(\frac{s}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{s}{2}-\frac{\nu}{2}\right). \tag{5.3}$$

Proof. Example 3.11 gives

$$\begin{aligned}
\int_0^\infty t^{s-1}K_\nu(t)\,dt &= \frac{\sqrt{\pi}}{2^\nu\Gamma(\nu+1/2)}\int_0^1 x^{-2\nu-1}(1-x^2)^{\nu-1/2}\int_0^\infty t^{\nu+s-1}e^{-t/x}\,dt\,dx \\
&= \frac{\sqrt{\pi}\,\Gamma(\nu+s)}{2^\nu\Gamma(\nu+1/2)}\int_0^1 x^{s-\nu-1}(1-x^2)^{\nu-1/2}\,dx \\
&= \frac{\sqrt{\pi}\,\Gamma(\nu+s)}{2^{\nu+1}\Gamma(\nu+1/2)}\int_0^1 u^{(s-\nu)/2-1}(1-u)^{\nu+1/2-1}\,du \\
&= \frac{\sqrt{\pi}\,\Gamma(\nu+s)\,\Gamma(\frac{s-\nu}{2})\Gamma(\nu+1/2)}{2^{\nu+1}\Gamma(\nu+1/2)\Gamma(\frac{s+\nu+1}{2})} \\
&= 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right).
\end{aligned}$$

□

An alternative proof is offered next.

Proof. Since $K_\nu = K_{-\nu}$, we may assume that $\nu \geqslant 0$. By the Mellin inversion formula, the evaluation (5.3) is equivalent to

$$K_\nu\left(ax\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{s-2}a^{-s}\Gamma\left(\frac{s}{2}+\frac{\nu}{2}\right)\Gamma\left(\frac{s}{2}-\frac{\nu}{2}\right)x^{-s}\,dx \tag{5.4}$$

where $c > \nu$. The integrand has poles at $s = \pm\nu - 2n$ for $n = 0, 1, 2, \ldots$. Assuming that $\nu \notin \mathbb{Z}$, all poles are of first order and the residue at $s = \pm\nu - 2n$ is $2\left(-1\right)^n/n!$. Closing the contour of (5.4) to the left and collecting the residues yields

$$\frac{1}{2}\sum_{n=0}^\infty \frac{(-1)^n}{n!}\left[\Gamma\left(\nu-n\right)\left(\frac{ax}{2}\right)^{-\nu+2n} + \Gamma\left(-\nu-n\right)\left(\frac{ax}{2}\right)^{\nu+2n}\right].$$

Using Euler's reflection formula in the form

$$\Gamma\left(\mu-n\right) = \frac{(-1)^n}{\Gamma\left(1-\mu+n\right)}\frac{\pi}{\sin\left(\pi\mu\right)},$$

this becomes

$$\frac{\pi}{2\sin\left(\pi\nu\right)}\sum_{n=0}^\infty\frac{1}{n!}\left[\frac{1}{\Gamma\left(1-\nu+n\right)}\left(\frac{ax}{2}\right)^{-\nu+2n} - \frac{1}{\Gamma\left(1+\nu+n\right)}\left(\frac{ax}{2}\right)^{\nu+2n}\right].$$

The definitions (1.6) and (1.7) show that this last term is $K_\nu(ax)$, as claimed. □

Example 5.2. Entry **3.389.4** of [**8**] is

$$\int_a^\infty x(x^2-a^2)^{\nu-1}e^{-\mu x}\,dx = \frac{2^{\nu-1/2}}{\sqrt{\pi}}\mu^{1/2-\nu}a^{\nu+1/2}\Gamma(\nu)K_{\nu+1/2}(a\mu), \tag{5.5}$$

where we have replaced the original parameter u in [**8**] by a in order to avoid confusion with the parameter μ. This identity is now verified.

Use the formula (5.2) to replace the term $e^{-\mu x}$ and reverse the order of integration to obtain

$$\int_a^\infty x(x^2-a^2)^{\nu-1}e^{-\mu x}\,dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mu^{-s}\Gamma(s)\left(\int_a^\infty x^{1-s}(x^2-a^2)^{\nu-1}\,dx\right)ds.$$

LEMMA 5.2. *The inner integral is given by*

$$\int_a^\infty x^{1-s}(x^2-a^2)^{\nu-1}\,dx = \frac{1}{\sqrt{\pi}\Gamma(s)}a^{2\nu-s}\Gamma\left(\frac{s}{2}-\nu\right)\Gamma(\nu)2^{s-2}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right). \tag{5.6}$$

PROOF. Let $x=at$ and $t=y^{-1/2}$ to produce

$$\begin{aligned}\int_a^\infty x^{1-s}(x^2-a^2)^{\nu-1}\,dx &= a^{2\nu-s}\int_1^\infty t^{1-s}(t^2-1)^{\nu-1}\,dt \\ &= \frac{1}{2}a^{2\nu-s}\int_0^1 y^{s/2-\nu-1}(1-y)^{\nu-1}\,dy \\ &= \frac{1}{2}a^{2\nu-s}B\left(s/2-\nu,\nu\right) \\ &= \frac{a^{2\nu-s}\Gamma(s/2-\nu)\,\Gamma(\nu)}{2\Gamma(s/2)}.\end{aligned}$$

Now employ the duplication formula for the gamma function

$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}}\Gamma(s)\Gamma(s+\tfrac{1}{2}) \tag{5.7}$$

to obtain the result. □

This produces

$$\int_a^\infty x(x^2-a^2)^{\nu-1}e^{-\mu x}\,dx = \frac{a^{2\nu}\Gamma(\nu)}{8\pi^{3/2}i}\int_{c-i\infty}^{c+i\infty}\left(\frac{a\mu}{2}\right)^{-s}\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}-\nu\right)ds.$$

The parameter ν is assumed to be real. Now shift the contour of integration by $z=s-\nu+\frac{1}{2}$ to obtain, with $c'=c-\nu+\frac{1}{2}$,

$$\int_a^\infty x(x^2-a^2)^{\nu-1}e^{-\mu x}\,dx =$$
$$\frac{\Gamma(\nu)}{\sqrt{\pi}}\left(\frac{2}{\mu}\right)^{\nu-1/2}a^{\nu+1/2}\int_{c'-i\infty}^{c'+i\infty}\left(\frac{a\mu}{2}\right)^{-z}\frac{1}{4}\Gamma\left(\frac{z}{2}+\frac{\nu+1/2}{2}\right)\Gamma\left(\frac{z}{2}-\frac{\nu+1/2}{2}\right)dz.$$

The result now follows from Lemma 5.1.

EXAMPLE 5.3. The special case $\nu=\frac{1}{2}$ of Example 5.2 is

$$\int_a^\infty \frac{xe^{-\mu x}\,dx}{\sqrt{x^2-a^2}} = aK_1(a\mu). \tag{5.8}$$

This appears as entry **3.365.2** of [**8**].

EXAMPLE 5.4. Entry **3.366.2** is

$$\int_0^\infty \frac{(x+\beta)\, e^{-\mu x}\, dx}{\sqrt{x^2+2\beta x}} = \beta e^{\beta\mu} K_1(\beta\mu). \tag{5.9}$$

To evaluate this result, let $t = x+\beta$ and use Example 5.3.

6. A family of integrals and a recurrence

Section **3.461** of [**8**] contains four entries that are part of the family

$$f_n(a,b) := \int_0^\infty x^{2n} \exp\left(-a\sqrt{x^2+b^2}\right)\, dx. \tag{6.1}$$

The evaluation of this family is discussed in this section.

The change of variables $t = \sqrt{x^2+b^2}$ produces

$$f_n(a,b) = \int_b^\infty t(t^2-b^2)^{n-\frac{1}{2}} e^{-at}\, dt. \tag{6.2}$$

This integral was evaluated in Example 5.2 as

$$f_n(a,b) = \frac{b}{\sqrt{\pi}}\Gamma(n+\tfrac{1}{2}) \left(\frac{2b}{a}\right)^n K_{n+1}(ab). \tag{6.3}$$

The example $n=0$ appears as entry **3.461.6** in the form

$$\int_0^\infty \exp\left(-a\sqrt{x^2+b^2}\right)\, dx = bK_1(ab). \tag{6.4}$$

The remaining examples of the stated family are simplified using the recurrence

$$K_\nu(z) = \frac{2(\nu-1)}{z}K_{\nu-1}(z) + K_{\nu-2}(z). \tag{6.5}$$

EXAMPLE 6.1. Entry **3.461.7** states that

$$f_1(a,b) = \int_0^\infty x^2 \exp\left(-a\sqrt{x^2+b^2}\right)\, dx = \frac{2b}{a^2}K_1(ab) + \frac{b^2}{a}K_0(ab). \tag{6.6}$$

The form given in (6.3) is

$$f_1(a,b) = \frac{b^2}{a}K_2(ab). \tag{6.7}$$

The recurrence (6.5) gives

$$K_2(ab) = \frac{2}{ab}K_1(ab) + K_0(ab) \tag{6.8}$$

which produces the result.

EXAMPLE 6.2. The same procedure used in Example 6.1 gives the evaluation of entry **3.461.8** as

$$f_2(a,b) = \int_0^\infty x^4 \exp\left(-a\sqrt{x^2+b^2}\right)\, dx = \frac{12b^2}{a^3}K_2(ab) + \frac{3b^3}{a^2}K_1(ab) \tag{6.9}$$

and entry **3.461.9** as

$$f_3(a,b) = \int_0^\infty x^6 \exp\left(-a\sqrt{x^2+b^2}\right)\,dx = \frac{90b^3}{a^4}K_3(ab) + \frac{15b^4}{a^3}K_2(ab). \tag{6.10}$$

Remark 6.1. The recurrence (6.5) converts the evaluation of $f_n(a,b)$ into an expression depending only upon $K_0(ab)$ and $K_1(ab)$. For instance,

$$f_2(a,b) = \frac{12b^2}{a^3}K_0(ab) + \left(\frac{24b}{a^4} + \frac{3b^3}{a^2}\right)K_1(ab) \tag{6.11}$$

and

$$f_3(a,b) = \left(\frac{360b^2}{a^5} + \frac{15b^4}{a^3}\right)K_0(ab) + \left(\frac{720b}{a^6} + \frac{120b^3}{a^4}\right)K_1(ab). \tag{6.12}$$

Experimentally we discovered that introducing the scaling

$$g_n(c) = \frac{a^{2n}2^n n!}{b\,(2n)!} f_n(a,b) \tag{6.13}$$

and label $c = ab$ and $x = K_0(c)$, $y = K_1(c)$, the expressions for the integrals simplify. The first few polynomials are

$$\begin{aligned}
g_3(c) &= c(c^2+24)x + 8(c^2+6)y \\
g_4(c) &= 12c(c^2+16)x + (c^4+72c^2+384)y \\
g_5(c) &= c(c^4+144c^2+1920)x + 6(3c4+128c^2+640)y \\
g_6(c) &= 24c(c^4+80c^2+960)x + (c^6+288c^4+9600c^2+46080)y.
\end{aligned}$$

Properties of the polynomials appearing in the coefficients will be reported elsewhere. For example, the function $g_n(c)$ satisfies the differential equation

$$b^2 g_n''(b) - (2n-1)b g_n'(b) - \left((ab)^2 + 2n + 1\right) g_n(b) = 0, \tag{6.14}$$

and also the recurrence

$$g_{n+2}(b) - 2(n+2)g_{n+1}(b) - (ab)^2 g_n(b) = 0. \tag{6.15}$$

7. A hyperexponential example

This section discusses several evaluations of entry **3.323.3**

$$\int_0^\infty \exp\left(-\beta^2x^4 - 2\gamma^2x^2\right)dx = 2^{-3/2}\frac{\gamma}{\beta}e^{\gamma^4/2\beta^2}\,K_{1/4}\left(\frac{\gamma^4}{2\beta^2}\right). \tag{7.1}$$

This example also appears as entry **3.469.1** in the form

$$\int_0^\infty \exp\left(-\mu x^4 - 2\nu x^2\right)dx = \frac{1}{4}\sqrt{\frac{2\nu}{\mu}}\exp\left(\frac{\nu^2}{2\mu}\right)\,K_{1/4}\left(\frac{\nu^2}{2\mu}\right). \tag{7.2}$$

The change of variables $x = \gamma t/\beta$ converts (7.1) into the form

$$\int_0^\infty e^{-2b(t^2+1)^2}\,dt = 2^{-3/2}e^{-b}K_{1/4}(b), \tag{7.3}$$

with $b = \gamma^4/2\beta^2$. A similar change of variables converts (7.2) to (7.3).

7.1. A traditional proof. Recall that K_ν is defined in terms of I_ν. The definition of I_ν as the series (1.7) is equivalent to the hypergeometric representation

$$\Gamma(\nu+1) I_\nu(x) = \left(\frac{x}{2}\right)^\nu {}_0F_1\left(\left.\begin{matrix} - \\ \nu+1 \end{matrix}\right| \frac{x^2}{4}\right). \tag{7.4}$$

Applying Kummer's second transformation, see for instance [Andrews-Askey-Roy, Section 4.1], to (7.4) one obtains

$$\Gamma(\nu+1) I_\nu(x) = \left(\frac{x}{2}\right)^\nu e^{-x} {}_1F_1\left(\left.\begin{matrix} \nu+\frac{1}{2} \\ 2\nu+1 \end{matrix}\right| 2x\right). \tag{7.5}$$

Consider the integral in (7.3). The change of variables $x = t^2$ followed by a series expansion and the further change of variables $s = x^2$ gives

$$\begin{aligned}
\int_0^\infty e^{-2b(t^2+1)^2}\,dt &= \frac{1}{2} e^{-2b} \int_0^\infty x^{-1/2} e^{-2bx^2-4bx}\,dx \\
&= \frac{1}{2} e^{-2b} \sum_{k=0}^\infty \frac{(-4b)^k}{k!} \int_0^\infty x^{k-1/2} e^{-2bx^2}\,dx \\
&= \frac{1}{4} e^{-2b} \sum_{k=0}^\infty \frac{(-4b)^k}{k!} \int_0^\infty s^{k/2-3/4} e^{-2bs}\,ds \\
&= \frac{1}{4} e^{-2b} \sum_{k=0}^\infty \frac{(-4b)^k}{k!} \frac{\Gamma(1/4+k/2)}{(2b)^{k/2+1/4}} \\
&= \frac{e^{-2b}}{4(2b)^{1/4}} \sum_{k=0}^\infty \frac{(-2\sqrt{2b})^k}{k!} \Gamma\left(\frac{k}{2}+\frac{1}{4}\right).
\end{aligned}$$

Writing the terms according to the parity of the index k produces

$$\int_0^\infty e^{-2b(t^2+1)^2}\,dt = \frac{e^{-2b}}{4(2b)^{1/4}} \left[\sum_{k=0}^\infty \frac{(8b)^k}{(2k)!} \Gamma\left(k+\frac{1}{4}\right) - 2\sqrt{2b} \sum_{k=0}^\infty \frac{(8b)^k}{(2k+1)!} \Gamma\left(k+\frac{3}{4}\right)\right].$$

Now use the definition of the Pochhammer symbol

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \tag{7.6}$$

to write

$$\Gamma\left(k+\tfrac{1}{4}\right) = \left(\tfrac{1}{4}\right)_k \Gamma\left(\tfrac{1}{4}\right), \quad \Gamma\left(k+\tfrac{3}{4}\right) = \left(\tfrac{3}{4}\right)_k \Gamma\left(\tfrac{3}{4}\right), \tag{7.7}$$

and

$$(2k)! = 2^{2k} \left(\tfrac{1}{2}\right)_k (1)_k, \quad (2k+1)! = 2^{2k} \left(\tfrac{3}{2}\right)_k (1)_k \tag{7.8}$$

to produce

$$\begin{aligned}
\int_0^\infty e^{-2b(t^2+1)^2}\,dt &= \frac{e^{-2b}}{4(2b)^{1/4}} \left[\Gamma\left(\frac{1}{4}\right) \sum_{k=0}^\infty \frac{(2b)^k}{k!} \frac{(1/4)_k}{(1/2)_k} - 2\sqrt{2b}\,\Gamma\left(\frac{3}{4}\right) \sum_{k=0}^\infty \frac{(2b)^k}{k!} \frac{(3/4)_k}{(3/2)_k}\right] \\
&= \frac{e^{-2b}}{4(2b)^{1/4}} \left\{\Gamma\left(\frac{1}{4}\right) {}_1F_1\left(\left.\begin{matrix} 1/4 \\ 1/2 \end{matrix}\right| 2b\right) - 2\sqrt{2b}\,\Gamma\left(\frac{3}{4}\right) {}_1F_1\left(\left.\begin{matrix} 3/4 \\ 3/2 \end{matrix}\right| 2b\right)\right\}.
\end{aligned}$$

Applying the representation (7.5) of I_ν gives

$$\int_0^\infty e^{-2b(t^2+1)^2}\,dt = \frac{\pi}{4}e^{-b}\left(I_{-1/4}(b) - I_{1/4}(b)\right). \tag{7.9}$$

This completes the traditional proof.

7.2. An automatic proof. This second proof of (7.1) is computer generated. The reader will find in [**11**] a selection of examples from [**8**] where similar computer generated proofs are described.

The condition $\operatorname{Re}\beta^2 > 0$, stated below, ensures convergence of the integral. Observe that the left-hand side of (7.10) is analytic in both γ and β, while the right-hand side needs to be interpreted such that it shares this analyticity. In order to not worry about taking the right branch-cuts on the right-hand side, we restrict to $\gamma \geqslant 0$ and $\beta > 0$. These conditions can then be removed at the end of the argument by analytic continuation.

THEOREM 7.1. *For complex γ, β such that* $\operatorname{Re}\left(\beta^2\right) > 0$*, we have*

$$F(\gamma) := \int_0^\infty \exp\left(-\beta^2x^4 - 2\gamma^2x^2\right)\mathrm{d}x = 2^{-3/2}\frac{\gamma}{\beta}\exp\left(\frac{\gamma^4}{2\beta^2}\right)K_{1/4}\left(\frac{\gamma^4}{2\beta^2}\right). \tag{7.10}$$

PROOF. Since the integrand is hyperexponential, we can apply the Almkvist-Zeilberger algorithm [**1**], which is a differential analogue to Zeilberger's celebrated summation algorithm for hypergeometric summands. These algorithms sometimes are also subsumed under the name WZ theory. In the following we denote the integrand by $f(x,\gamma) := \exp\left(-\beta^2x^4 - 2\gamma^2x^2\right)$. Using creative telescoping one finds that

$$\left(A + D_x \cdot 4\gamma^3 x\right)\cdot f(x,\gamma) = 0 \tag{7.11}$$

where $A := \beta^2\gamma D_\gamma^2 - \left(4\gamma^4 + \beta^2\right)D_\gamma - 4\gamma^3$ and $D_x = \frac{d}{dx}$, $D_\gamma = \frac{d}{d\gamma}$. Hence it follows that

$$\begin{aligned} A\cdot\int_0^T f(x,\gamma)\,dx &= \int_0^T A\cdot f(x,\gamma)\,dx \\ &= -\int_0^T D_x\cdot 4\gamma^3x\cdot f(x,\gamma)\,dx \\ &= -4\gamma^3T\cdot f(T,\gamma). \end{aligned}$$

In the limit $T\to\infty$, we therefore have

$$A\cdot\int_0^\infty f(x,\gamma)\,\mathrm{d}x = 0.$$

Let $G(\gamma)$ be the right-hand side of (7.10). In the light of the differential equation (1.5) satisfied by the modified Bessel function $K_{1/4}$, a direct calculation shows that $G(\gamma)$ is also annihilated by A, that is

$$A\cdot G(\gamma) = A\cdot 2^{-3/2}\frac{\gamma}{\beta}\exp\left(\frac{\gamma^4}{2\beta^2}\right)K_{1/4}\left(\frac{\gamma^2}{2\beta^2}\right) = 0.$$

Thus the claim follows by checking that $F(0) = G(0)$ and $F'(0) = G'(0)$. The explicit evaluations

$$F(0) = \int_0^\infty \exp\left(-\beta^2 x^4\right)\, dx = \frac{\Gamma(1/4)}{4\sqrt{\beta}}$$
$$F'(0) = \left[-4\gamma \int_0^\infty x^2 \exp\left(-\beta^2 x^4 - 2\gamma^2 x^2\right)\, dx\right]_{\gamma=0} = 0$$

confirm that these values agree with $G(0)$ and $G'(0)$. ☐

Remark 7.1. It remains to explain how the relation (7.11) can be found using the Mathematica package `HolonomicFunctions` [**10**]. After loading the package, one just has to type:

In[1]:= $\mathbf{CreativeTelescoping[Exp[-\boldsymbol{b}\verb|^|2 * \boldsymbol{x}\verb|^|4 - 2 * \boldsymbol{c}\verb|^|2 * \boldsymbol{x}\verb|^|2], Der[\boldsymbol{x}], Der[\boldsymbol{c}]]}$

Out[1]= $\{\{b^2 c D_c^2 + (-b^2 - 4c^4) D_c - 4c^3\}, \{4c^3 x\}\}$

Remark 7.2. Instead of to (7.1), the creative telescoping approach can also be applied to (7.3). However, in that case, the task of comparing initial values is not so simple, as the integral (7.3) does not converge for $b = 0$. As a solution one could compute the initial values at $b = 1$ but the resulting integrals are not trivial themselves.

7.3. An evaluation by the method of brackets. This method was developed by I. Gonzalez and I. Schmidt in [**7**] in the context of definite integrals coming from Feynman diagrams. The complete operational rules are described in [**5, 6**]. Even though this is a formal method for integration, some of the rules have been made rigorous in [**2**]. A code has been produced in [**9**].

The basic idea is to associate a *bracket* to the divergent integral

$$\langle a \rangle = \int_0^\infty x^{a-1}\, dx. \tag{7.12}$$

This extends to the integral of a function expanded in power series: let f be a formal power series

$$f(x) = \sum_{n=0}^\infty a_n x^{\alpha n + \beta - 1}. \tag{7.13}$$

The symbol

$$\int_0^\infty f(x)\, dx \doteq \sum_n a_n \langle \alpha n + \beta \rangle \tag{7.14}$$

represents a *bracket series* assignement to the integral on the left. Rule 7.2 describes how to evaluate this series.

The symbol

$$\phi_n = \frac{(-1)^n}{\Gamma(n+1)} \tag{7.15}$$

will be called the *indicator of* n, it gives a simpler form for the bracket series associated to an integral. For example,

$$\int_0^\infty x^{a-1}e^{-x}\,dx \doteq \sum_n \phi_n \langle n+a\rangle. \tag{7.16}$$

The integral is the gamma function $\Gamma(a)$ and the right-hand side its bracket expansion.

RULE 7.1. For $\alpha \in \mathbb{C}$, the expression

$$(a_1+a_2+\cdots+a_r)^\alpha \tag{7.17}$$

is assigned the bracket series

$$\sum_{m_1,\cdots,m_r} \phi_{1,2,\cdots,r}\, a_1^{m_1}\cdots a_r^{m_r}\frac{\langle -\alpha+m_1+\cdots+m_r\rangle}{\Gamma(-\alpha)}, \tag{7.18}$$

where $\phi_{1,2,\cdots,r}$ is a short-hand notation for the product $\phi_{m_1}\phi_{m_2}\cdots\phi_{m_r}$.

RULE 7.2. The series of brackets

$$\sum_n \phi_n f(n)\langle an+b\rangle \tag{7.19}$$

is given the *value*

$$\frac{1}{a}f(n^*)\Gamma(-n^*) \tag{7.20}$$

where n^* solves the equation $an+b=0$.

RULE 7.3. A two-dimensional series of brackets

$$\sum_{n_1,n_2} \phi_{n_1,n_2} f(n_1,n_2)\langle a_{11}n_1+a_{12}n_2+c_1\rangle\langle a_{21}n_1+a_{22}n_2+c_2\rangle \tag{7.21}$$

is assigned the *value*

$$\frac{1}{|a_{11}a_{22}-a_{12}a_{21}|}f(n_1^*,n_2^*)\Gamma(-n_1^*)\Gamma(-n_2^*) \tag{7.22}$$

where (n_1^*,n_2^*) is the unique solution to the linear system

$$\begin{aligned} a_{11}n_1+a_{12}n_2+c_1 &= 0,\\ a_{21}n_1+a_{22}n_2+c_2 &= 0, \end{aligned} \tag{7.23}$$

obtained by the vanishing of the expressions in the brackets. A similar rule applies to higher dimensional series, that is,

$$\sum_{n_1}\cdots\sum_{n_r}\phi_{1,\cdots,r}f(n_1,\cdots,n_r)\langle a_{11}n_1+\cdots a_{1r}n_r+c_1\rangle\cdots\langle a_{r1}n_1+\cdots a_{rr}n_r+c_r\rangle$$

is assigned the value

$$\frac{1}{|\det(A)|}f(n_1^*,\cdots,n_r^*)\Gamma(-n_1^*)\cdots\Gamma(-n_r^*), \tag{7.24}$$

where A is the matrix of coefficients (a_{ij}) and $\{n_i^*\}$ is the solution of the linear system obtained by the vanishing of the brackets. The value is not defined if the matrix A is not invertible.

RULE 7.4. In the case where the assignment leaves free parameters, any divergent series in these parameters is discarded. In case several choices of free parameters are available, the series that converge in a common region are added to contribute to the integral.

The method of brackets is now employed to verify (7.1) in its original form

$$\int_0^\infty \exp\left(-\beta^2x^4 - 2\gamma^2x^2\right)\,dx = 2^{-\frac{3}{2}}\frac{\gamma}{\beta}e^{\frac{\gamma^4}{2\beta^2}}K_{1/4}\left(\frac{\gamma^4}{2\beta^2}\right).$$

Start with the bracket-series

$$\begin{aligned}\int_0^\infty e^{-(\beta^2x^4+2\gamma^2x^2)}\,dx &= \int_0^\infty \sum_{n_1}\phi_{n_1}(\beta^2x^4+2\gamma^2x^2)^{n_1}\,dx\\ &= \int_0^\infty \sum_{n_1}\phi_{n_1}x^{2n_1}(\beta^2x^2+2\gamma^2)^{n_1}\,dx\end{aligned}$$

and expand the term $(\beta^2x^2+2\gamma^2)^{n_1}$ in a double bracket series to obtain

$$\begin{aligned}\int_0^\infty e^{-(\beta^2x^4+2\gamma^2x^2)}\,dx &= \int_0^\infty \sum_{n_1}\phi_{n_1}x^{2n_1}\left(\sum_{n_2}\sum_{n_3}\phi_{n_2}\phi_{n_3}(\beta^2x^2)^{n_2}(2\gamma^2)^{n_3}\frac{\langle -n_1+n_2+n_3\rangle}{\Gamma(-n_1)}\right)dx\\ &= \sum_{n_1}\sum_{n_2}\sum_{n_3}\phi_{n_1}\phi_{n_2}\phi_{n_3}\frac{2^{n_3}\beta^{2n_2}\gamma^{2n_3}}{\Gamma(-n_1)}\langle 2n_1+2n_2+1\rangle\langle -n_1+n_2+n_3\rangle\end{aligned}$$

The result is a 3-dimensional sum with two brackets. The rules state that the integral is now expressed as a single sum in the free parameter coming from solving the system

$$\begin{aligned}2n_1+2n_2+1 &= 0\\ -n_1+n_2+n_3 &= 0.\end{aligned}$$

The system is of rank 2, so there are three cases to consider according to the choice of the free parameter.

Case 1: n_1 free: the resulting system is

$$\begin{aligned}2n_2 &= -2n_1-1\\ n_2+n_3 &= n_1,\end{aligned}$$

and the corresponding matrix has $\det(A) = -2$. The solutions are $n_3^* = 2n_1+\frac{1}{2}$ and $n_2^* = -n_1-\frac{1}{2}$. The resulting sum is

$$\sum_{n_1}\frac{(-1)^{n_1}2^{2n_1-1/2}\beta^{-2n_1-1}\gamma^{4n_1+1}\Gamma(-2n_1-1/2)\Gamma(n_1+1/2)}{\Gamma(n_1+1)\Gamma(-n_1)}$$

and it vanishes due to the presence of $\Gamma(-n_1)$ in the denominator.

Case 2: n_2 free: in this case the matrix of coefficients satisfies $\det(A) = 2$ and the solutions are $n_1^* = -n_2 - \frac{1}{2}$ and $n_3^* = -2n_2 - \frac{1}{2}$. The resulting sum

$$\sum_{n_2} \frac{(-1)^{n_2} 2^{-2n_2-3/2} \beta^{2n_2} \gamma^{-4n_2-1} \Gamma(2n_2 + \frac{1}{2})}{\Gamma(n_2+1)}$$

is divergent, so it is discarded.

Case 3: n_3 free: then $\det(A) = 4$ and $n_1^* = \frac{1}{2}n_3 - \frac{1}{4}$ and $n_2^* = -\frac{1}{2}n_3 - \frac{1}{4}$. The corresponding series is

$$\sum_{n_3} \frac{(-1)^{n_3} 2^{n_3-2} \beta^{-n_3-1/2} \gamma^{2n_3} \Gamma(n_3/2 + 1/4)}{\Gamma(n_3+1)} = \frac{1}{4\sqrt{\beta}} \sum_{n_3} (-1)^{n_3} \delta^{n_3} \frac{\Gamma(\frac{1}{2}n_3 + \frac{1}{4})}{\Gamma(n_3+1)},$$

with $\delta = 2\gamma^2/\beta$. In order simplify the result split the sum according to the parity of n_3 to produce

$$S := \frac{1}{4\sqrt{\beta}} \sum_{n=0}^{\infty} \delta^{2n} \frac{\Gamma(n+\frac{1}{4})}{\Gamma(2n+1)} - \frac{1}{4\sqrt{\beta}} \sum_{n=0}^{\infty} \delta^{2n+1} \frac{\Gamma(n+\frac{3}{4})}{\Gamma(2n+2)}.$$

Now use (7.7) and (7.8) to produce

$$S = \frac{1}{4\sqrt{\beta}} \left\{ \Gamma\left(\frac{1}{4}\right) {}_1F_1\left(\left. \begin{matrix} 1/4 \\ 1/2 \end{matrix} \right| \frac{\delta^2}{4} \right) - \delta \Gamma\left(\frac{3}{4}\right) {}_1F_1\left(\left. \begin{matrix} 3/4 \\ 3/2 \end{matrix} \right| \frac{\delta^2}{4} \right) \right\}.$$

The claim is thus seen to be equivalent to the identity

$$\Gamma\left(\frac{1}{4}\right) {}_1F_1\left(\left. \begin{matrix} 1/4 \\ 1/2 \end{matrix} \right| b \right) - 2\sqrt{b}\Gamma\left(\frac{3}{4}\right) {}_1F_1\left(\left. \begin{matrix} 3/4 \\ 3/2 \end{matrix} \right| b \right) = \sqrt{2} b^{1/4} e^{b/2} K_{1/4}\left(\frac{b}{2}\right),$$

where $b = \delta^2/4$. The identity to be established is now expressed in terms of the Bessel function I_ν using (1.7). The result is

$$\Gamma\left(\frac{1}{4}\right) {}_1F_1\left(\left. \begin{matrix} 1/4 \\ 1/2 \end{matrix} \right| b \right) - 2\sqrt{b}\Gamma\left(\frac{3}{4}\right) {}_1F_1\left(\left. \begin{matrix} 3/4 \\ 3/2 \end{matrix} \right| b \right) = \pi b^{1/4} e^{b/2} \left(I_{-1/4}\left(\frac{b}{2}\right) - I_{1/4}\left(\frac{b}{2}\right) \right).$$

Using the expansion (1.6) shows that the right-hand side of the previous expression is $\pi e^{b/2}$ times the series

$$\sum_{k=0}^{\infty} \frac{1}{\Gamma(k+3/4)k!} \frac{b^{2k}}{2^{4k-1/2}} - \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+5/4)k!} \frac{b^{2k+1/2}}{2^{4k+1/2}}.$$

Each of these series can be simplified. Introduce $c = b^2/16$ and write

$$\sum_{k=0}^{\infty} \frac{1}{\Gamma(k+3/4)k!} \frac{b^{2k}}{2^{4k-1/2}} = \frac{\sqrt{2}}{\Gamma(3/4)} \sum_{k=0}^{\infty} \frac{1}{(3/4)_k} \frac{c^k}{k!} = \frac{\sqrt{2}}{\Gamma(3/4)} {}_0F_1\left(\left. \begin{matrix} - \\ 3/4 \end{matrix} \right| c \right)$$

and

$$\sum_{k=0}^{\infty} \frac{1}{\Gamma(k+5/4)k!} \frac{b^{2k+1/2}}{2^{4k+1/2}} = \frac{\sqrt{b}}{\sqrt{2}\Gamma(5/4)} \sum_{k=0}^{\infty} \frac{1}{(5/4)_k} \frac{c^k}{k!} = \frac{\sqrt{b}}{\sqrt{2}\Gamma(5/4)} {}_0F_1\left(\left. \begin{matrix} - \\ 5/4 \end{matrix} \right| c \right)$$

The proof of the main identity (7.1) by the method of brackets is now reduced to verifying

$$\Gamma\left(\frac{1}{4}\right) {}_1F_1\left(\left.\begin{matrix}1/4\\1/2\end{matrix}\right| b\right) - 2\sqrt{b}\,\Gamma\left(\frac{3}{4}\right)\ {}_1F_1\left(\left.\begin{matrix}3/4\\3/2\end{matrix}\right| b\right) = \pi e^{b/2}\left\{\frac{\sqrt{2}}{\Gamma(3/4)}\ {}_0F_1\left(\left.\begin{matrix}-\\3/4\end{matrix}\right| c\right) - {}_0F_1\left(\left.\begin{matrix}-\\5/4\end{matrix}\right| c\right)\right\}. \tag{7.25}$$

The exponents appearing in the series above are either integers or $\frac{1}{2}$ plus an integer. Matching these two types separately shows that the main evaluation follows from the identities

$${}_1F_1\left(\left.\begin{matrix}1/4\\1/2\end{matrix}\right| b\right) = e^{b/2}{}_0F_1\left(\left.\begin{matrix}-\\3/4\end{matrix}\right| \frac{b^2}{16}\right) \text{ and } {}_1F_1\left(\left.\begin{matrix}3/4\\3/2\end{matrix}\right| b\right) = e^{b/2}{}_0F_1\left(\left.\begin{matrix}-\\5/4\end{matrix}\right| \frac{b^2}{16}\right).$$

These are special cases of Kummer's second transformation which is exhibited in the equivalence of (7.4) and (7.5). This completes the proof of Example 7.1.

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Received May 15, 2012, revised October 20, 2012

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