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The integrals in Gradshteyn and Ryzhik. Part 26: The exponential integral

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ABSTRACT. The table of Gradshteyn and Ryzhik contains many entries where the evaluation is given in terms of the exponential integral. A selection of these formulas are established.

1. Introduction

The *exponential integral* function is defined by

$$(1.1) \quad \text{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t} dt$$

for $x < 0$. In the case $x > 0$ we use the Cauchy principal value

$$(1.2) \quad \text{Ei}(x) = - \lim_{\epsilon \rightarrow 0^+} \left[\int_{-x}^{-\epsilon} \frac{e^{-t}}{t} dt + \int_{\epsilon}^{\infty} \frac{e^{-t}}{t} dt \right].$$

This appears as entry **3.351.6** in [2]

Another function defined by an integral is the *logarithmic integral*:

$$(1.3) \quad \text{li}(u) := \int_0^u \frac{dx}{\ln x}.$$

This is entry **4.211.2**. The change of variables $t = \ln x$ shows that

$$(1.4) \quad \text{li}(u) = \text{Ei}(\ln u).$$

Observe that the integral defining li diverges as $u \rightarrow \infty$. Indeed, entry **4.211.1** states that

$$(1.5) \quad \int_e^{\infty} \frac{dx}{\ln x} = +\infty$$

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This is evident from the change of variables $t = \ln x$ that yields

$$(1.6) \quad \int_e^\infty \frac{dx}{\ln x} = \int_1^\infty \frac{e^t dt}{t} \geq \int_1^\infty \frac{dt}{t} = \infty.$$

2. Some simple changes of variables

The change of variables $t = -as$ yields

$$(2.1) \quad \int_{-x/a}^\infty \frac{e^{-as}}{s} ds = -\text{Ei}(x).$$

Replacing x by ax , this gives

$$(2.2) \quad \int_{-ax}^\infty \frac{e^{-t}}{t} dt = -\text{Ei}(ax).$$

The special choice $x = -a$ in (2.1) yields entry **3.351.5**:

$$(2.3) \quad \int_1^\infty \frac{e^{-as}}{s} ds = -\text{Ei}(-a).$$

The expression

$$(2.4) \quad \text{Ei}(-a) = - \int_1^\infty \frac{e^{-as}}{s} ds$$

is an analytic function of a for $\text{Re } a > 0$. This provides an analytic extension of $\text{Ei}(z)$ to the left half plane $\text{Re } z < 0$. Several entries of [2] are derived from here.

Example 2.1. For any β such that $u + \beta > 0$

$$(2.5) \quad \text{Ei}(-au - a\beta) = \text{Ei}(-a(u + \beta)) = - \int_{u+\beta}^\infty \frac{e^{-ax}}{x} dx$$

and then the shift $x \mapsto x + \beta$ produces

$$(2.6) \quad \text{Ei}(-au - a\beta) = -e^{-a\beta} \int_u^\infty \frac{e^{-ax}}{x + \beta} dx$$

that can be written as

$$(2.7) \quad \int_u^\infty \frac{e^{-ax}}{x + \beta} dx = -e^{a\beta} \text{Ei}(-au - a\beta).$$

This appears as entry **3.352.2**. This representation is valid or $\beta \in \mathbb{C}$ outside the half-line $(-\infty, u]$.

Example 2.2. The special case $u = 0$ and $\beta \notin (-\infty, 0]$ gives

$$(2.8) \quad \int_0^\infty \frac{e^{-ax}}{x + \beta} dx = -e^{a\beta} \text{Ei}(-a\beta).$$

This is entry **3.352.4** in [2].

Example 2.3. The difference of (2.7) and (2.8) produces

$$(2.9) \quad \int_0^u \frac{e^{-ax}}{x+\beta} dx = e^{au} [\text{Ei}(-au - a\beta) - \text{Ei}(-a\beta)].$$

This is entry **3.352.1**.

Example 2.4. Entry **3.352.3** states that

$$(2.10) \quad \int_u^v \frac{e^{-ax}}{x+\beta} dx = e^{a\beta} [\text{Ei}(-a(v+\beta)) - \text{Ei}(-a(u+\beta))].$$

This comes directly from (2.7):

$$(2.11) \quad \begin{aligned} \int_u^v \frac{e^{-ax} dx}{x+\beta} &= \int_u^\infty \frac{e^{-ax} dx}{x+\beta} - \int_v^\infty \frac{e^{-ax} dx}{x+\beta} \\ &= -e^{a\beta} \text{Ei}(-au - a\beta) + e^{a\beta} \text{Ei}(-av - a\beta). \end{aligned}$$

This is the result.

Example 2.5. In the expression (2.7), when $u > 0$, the parameter β may be taken in the range $\beta < u$, so that $x - \beta > 0$ for all $x \geq u$. This produces entry **3.352.5**

$$(2.12) \quad \int_u^\infty \frac{e^{-ax} dx}{x-\beta} = -e^{-a\beta} \text{Ei}(-a(u-\beta)).$$

Example 2.6. In the case $u = 0$ and $\beta < 0$, the entry in Example 2.5 can be written as

$$(2.13) \quad \int_0^\infty \frac{e^{-ax} dx}{\beta - x} = e^{-a\beta} \text{Ei}(a\beta).$$

This is entry **3.352.6** in [2].

3. Entries obtained by differentiation

This section presents proofs of some entries in [2] obtained by manipulations of derivatives of the exponential integral function.

Example 3.1. Entry **3.353.3** is

$$(3.1) \quad \int_0^\infty \frac{e^{-ax} dx}{(x+\beta)^2} = \frac{1}{\beta} + ae^{-a\beta} \text{Ei}(-a\beta).$$

To establish this, differentiate (2.7) and use

$$(3.2) \quad \frac{d}{dt} \text{Ei}(u) = \frac{e^u}{u} \frac{du}{dt}$$

to obtain

$$(3.3) \quad \int_u^\infty \frac{e^{-ax} dx}{(x+\beta)^2} = \frac{e^{-au}}{u+\beta} + ae^{a\beta} \text{Ei}(-au - a\beta).$$

The choice $u = 0$ with $\text{Re } \beta > 0$ and $\text{Re } a > 0$ gives the result.

Example 3.2. Entry **3.353.1** states that

$$(3.4) \quad \int_u^\infty \frac{e^{-ax} dx}{(x+\beta)^n} = e^{-au} \sum_{k=1}^{n-1} \frac{(k-1)!(-a)^{n-k-1}}{(n-1)!(u+\beta)^k} - \frac{(-a)^{n-1}}{(n-1)!} e^{a\beta} \text{Ei}(-au - a\beta).$$

can be easily established by induction. The initial step $n = 2$ is (3.3). Simply differentiate (3.4) with respect to β to move from n to $n + 1$. The details are left to the reader.

Example 3.3. The special case $u = 0$ of (3.4) gives

$$(3.5) \quad \int_0^\infty \frac{e^{-ax} dx}{(x+\beta)^n} = \sum_{k=1}^{n-1} \frac{(k-1)!(-a)^{n-k-1}}{(n-1)!\beta^k} - \frac{(-a)^{n-1}}{(n-1)!} e^{a\beta} \text{Ei}(-a\beta).$$

This is entry **3.353.2** in [2].

Example 3.4. Entry **3.351.4** states that

$$(3.6) \quad \int_u^\infty \frac{e^{-ax} dx}{x^{n+1}} = e^{-au} \sum_{k=1}^n \frac{(k-1)!(-a)^{n-k}}{n!u^k} + (-1)^{n+1} \frac{a^n}{n!} \text{Ei}(-au).$$

This result follows directly from (3.4) by taking $\beta = 0$ and $u > 0$ and then replacing n by $n + 1$. Changing the index of summation $k \mapsto n - k$, this may be written as it appears in [2]

$$(3.7) \quad \int_u^\infty \frac{e^{-ax} dx}{x^{n+1}} = \frac{e^{-au}}{u^n} \sum_{k=1}^n \frac{(-1)^k a^k u^k}{n(n-1)\cdots(n-k)} + (-1)^{n+1} \frac{a^n}{n!} \text{Ei}(-au).$$

Example 3.5. Entry **3.353.5** states that

$$(3.8) \quad \int_0^\infty \frac{x^n e^{-ax}}{x+\beta} dx = (-1)^{n-1} \beta^n e^{a\beta} \text{Ei}(-a\beta) + \sum_{k=1}^n (k-1)!(-\beta)^{n-k} \mu^{-k}.$$

In the special case $n = 1$, this reduces to

$$(3.9) \quad \int_0^\infty \frac{x e^{-ax}}{x+\beta} dx = \beta e^{a\beta} \text{Ei}(-a\beta) + \frac{1}{a}$$

which follows by differentiating (2.8) with respect to a . The general formula (3.8) is obtained directly by further differentiation.

Note 3.6. The entry **3.353.4**

$$(3.10) \quad \int_0^1 \frac{x e^x dx}{(x+1)^2} = \frac{e}{2} - 1,$$

which does not involve the exponential integral function, can be evaluated by simply integration by parts. This entry has been included in Section 10 of [1].

4. Entries with quadratic denominators

This section considers the entries in [2] where the integrand is an exponential term divided by a quadratic polynomial.

Example 4.1. Entry 3.354.3 is

$$(4.1) \quad \int_0^\infty \frac{e^{-ax} dx}{\beta^2 - x^2} = \frac{1}{2\beta} [e^{-a\beta} \text{Ei}(a\beta) - e^{a\beta} \text{Ei}(-a\beta)].$$

To evaluate this integral, assume $\beta \notin \mathbb{R}$ and use the partial fraction decomposition

$$(4.2) \quad \frac{1}{\beta^2 - x^2} = \frac{1}{2\beta} \left(\frac{1}{\beta - x} - \frac{1}{\beta + x} \right)$$

to obtain

$$(4.3) \quad \int_0^\infty \frac{e^{-ax} dx}{\beta^2 - x^2} = \frac{1}{2\beta} \left(\int_0^\infty \frac{e^{-ax} dx}{\beta - x} + \int_0^\infty \frac{e^{-ax} dx}{\beta + x} \right)$$

and now the result comes from (2.8) and (2.13). For $\beta \in \mathbb{R}$ the results valid as a Cauchy principal value integral.

Example 4.2. Differentiate (4.1) with respect to a produces

$$(4.4) \quad \int_0^\infty \frac{xe^{-ax} dx}{\beta^2 - x^2} = \frac{1}{2} [e^{-a\beta} \text{Ei}(a\beta) - e^{a\beta} \text{Ei}(-a\beta)].$$

This appears as entry 3.354.4 in [2].

Example 4.3. Entry 3.354.1

$$(4.5) \quad \int_0^\infty \frac{e^{-ax} dx}{\beta^2 + x^2} = \frac{1}{\beta} [\text{ci}(a\beta) \sin a\beta - \text{si}(a\beta) \cos a\beta]$$

involves the cosine and sine integrals defined by

$$(4.6) \quad \text{ci}(u) = - \int_u^\infty \frac{\cos t}{t} dt \text{ and } \text{si}(u) = - \int_u^\infty \frac{\sin t}{t} dt.$$

Start by replacing β by $i\beta$ in (4.1) to obtain

$$(4.7) \quad \int_0^\infty \frac{e^{-ax} dx}{\beta^2 + x^2} = \frac{1}{2i\beta} [e^{ia\beta} \text{Ei}(-ia\beta) - e^{-ia\beta} \text{Ei}(ia\beta)].$$

The classical identity of Euler

$$(4.8) \quad e^{\pm i\beta} = \cos a\beta \pm i \sin a\beta$$

gives the relation

$$(4.9) \quad \text{Ei}(\pm ia\beta) = \text{ci}(a\beta) \pm i \text{si}(a\beta).$$

Replacing in (4.7) gives the result.

Example 4.4. Differentiation of the entry in Example 4.3 gives

$$(4.10) \quad \int_0^\infty \frac{xe^{-ax} dx}{\beta^2 + x^2} = -\text{ci}(a\beta) \sin a\beta - \text{si}(a\beta) \cos a\beta.$$

This is entry 3.354.2 in [2].

The entries in Sections **3.355** and **3.356** are obtained by differentiation of the entries in Section **3.354** given above.

Example 4.5. Entry **3.355.1** is

$$(4.11) \quad \int_0^\infty \frac{e^{-ax} dx}{(\beta^2 + x^2)^2} = \frac{1}{2\beta^2} \{ \text{ci}(a\beta) \sin(a\beta) - \text{si}(a\beta) \cos(a\beta) - a\beta [\text{ci}(a\beta) \cos(a\beta) + \text{si}(a\beta) \sin(a\beta)] \}.$$

This is obtained by differentiation of Entry **3.354.1** given in (4.5).

Example 4.6. Entry **3.355.2** is

$$(4.12) \quad \int_0^\infty \frac{x e^{-ax} dx}{(\beta^2 + x^2)^2} = \frac{1}{2\beta^2} [1 - a\beta (\text{ci}(a\beta) \sin(a\beta) - \text{si}(a\beta) \cos(a\beta))].$$

This entry appeared with a typo in [2]. This entry is obtained by direct differentiation of (4.11).

Example 4.7. Differentiation of entries **3.354.3** and **3.354.4** produce

$$(4.13) \quad \int_0^\infty \frac{e^{-ax} dx}{(\beta^2 - x^2)^2} = \frac{1}{4\beta^3} [(a\beta - 1)e^{a\beta} \text{Ei}(-a\beta) + (1 + a\beta)e^{-a\beta} \text{Ei}(a\beta)]$$

and

$$(4.14) \quad \int_0^\infty \frac{x e^{-ax} dx}{(\beta^2 - x^2)^2} = \frac{1}{4\beta^2} [-2 + a\beta (e^{-a\beta} \text{Ei}(a\beta) - e^{a\beta} \text{Ei}(-a\beta))].$$

These are entries **3.355.3** and **3.355.4**, respectively.

Example 4.8. Differentiating (4.5) $2n$ -times with respect to a , gives

$$(4.15) \quad \int_0^\infty \frac{x^{2n} e^{-ax} dx}{\beta^2 + x^2} = (-1)^{n-1} \beta^{2n} [\text{ci}(a\beta) \cos(a\beta) + \text{si}(a\beta) \sin(a\beta)] + \frac{1}{\beta^{2n}} \sum_{k=1}^n (2n - 2k + 1)! (-a^2 \beta^2)^{k-1}.$$

This appears as Entry **3.356.2**. The identity

$$(4.16) \quad \int_0^\infty \frac{x^{2n} e^{-ax} dx}{\beta^2 - x^2} = \frac{1}{2} \beta^{2n-1} [e^{-a\beta} \text{Ei}(a\beta) - e^{a\beta} \text{Ei}(-a\beta)] - \frac{1}{\beta^{2n-1}} \sum_{k=1}^n (2n - 2k)! (a^2 \beta^2)^{k-1}$$

is obtained by differentiating (4.1). This appears as Entry **3.356.4**.

Example 4.9. The entries **3.356.1**

$$(4.17) \quad \int_0^\infty \frac{x^{2n+1} e^{-ax} dx}{\beta^2 + x^2} = (-1)^{n-1} \beta^{2n} [\text{ci}(a\beta) \cos a\beta + \text{si}(a\beta) \sin a\beta] + \frac{1}{a^{2n}} \sum_{k=1}^n (2n - 2k + 1)! (-a^2 \beta^2)^{k-1}$$

and entry **3.356.3**

$$(4.18) \quad \int_0^\infty \frac{x^{2n+1} e^{-ax} dx}{\beta^2 - x^2} = \frac{1}{2} \beta^{2n} [e^{a\beta} \text{Ei}(-a\beta) + e^{-a\beta} \text{Ei}(a\beta)] \\ - \frac{1}{a^{2n}} \sum_{k=1}^n (2n - 2k + 1)! (a^2 \beta^2)^{k-1}$$

are obtained by differentiating the entries in Example 4.8.

5. Some higher degree denominators

This section evaluates a series of entries in [2] where the integrand is an exponential times a rational function with denominator of degree larger than 2.

Example 5.1. Entry **3.358.1** is

$$(5.1) \quad \int_0^\infty \frac{e^{-ax} dx}{\beta^4 - x^4} = \\ \frac{1}{4\beta^3} \{e^{-a\beta} \text{Ei}(a\beta) - e^{a\beta} \text{Ei}(-a\beta) + 2 \text{ci}(a\beta) \sin(a\beta) - 2 \text{si}(a\beta) \cos(a\beta)\}$$

Start with the partial fraction decomposition

$$(5.2) \quad \frac{1}{\beta^4 - x^4} = \frac{1}{2\beta^2} \left(\frac{1}{\beta^2 - x^2} + \frac{1}{\beta^2 + x^2} \right)$$

which shows that the integral in question is a combination of (4.1) and (4.5). The result follows from here.

Example 5.2. Entry **3.358.2**

$$(5.3) \quad \int_0^\infty \frac{x e^{-ax} dx}{\beta^4 - x^4} = \\ \frac{1}{4\beta^2} \{e^{a\beta} \text{Ei}(-a\beta) + e^{-a\beta} \text{Ei}(a\beta) - 2 \text{ci}(a\beta) \cos(a\beta) - 2 \text{si}(a\beta) \sin(a\beta)\}.$$

This is obtained by differentiation of (5.1). The entries **3.358.3**

$$(5.4) \quad \int_0^\infty \frac{x^2 e^{-ax} dx}{\beta^4 - x^4} = \\ \frac{1}{4\beta} \{e^{-a\beta} \text{Ei}(a\beta) - e^{a\beta} \text{Ei}(-a\beta) - 2 \text{ci}(a\beta) \sin(a\beta) + 2 \text{si}(a\beta) \cos(a\beta)\}$$

and **3.358.4**

$$(5.5) \quad \int_0^\infty \frac{x^3 e^{-ax} dx}{\beta^4 - x^4} = \\ \frac{1}{4} \{e^{a\beta} \text{Ei}(-a\beta) + e^{-a\beta} \text{Ei}(a\beta) + 2 \text{ci}(a\beta) \cos(a\beta) + 2 \text{si}(a\beta) \sin(a\beta)\}$$

come from further differentiation.

The entries in Section **3.357** can be established by algebraic manipulations of the examples given above.

Example 5.3. Entry **3.357.1** states that

$$(5.6) \quad \int_0^\infty \frac{e^{-ax} dx}{\beta^3 + \beta^2 x + \beta x^2 + x^3} = \frac{1}{2\beta^2} \{ \text{ci}(a\beta)(\sin a\beta + \cos(a\beta)) + \text{si}(a\beta)(\sin a\beta - \cos(a\beta)) - e^{a\beta} \text{Ei}(-a\beta) \}$$

This formula is obtained from (5.1) and (5.3) and the algebraic identity

$$(5.7) \quad \frac{1}{\beta^3 + \beta^2 x + \beta x^2 + x^3} = \frac{\beta - x}{\beta^4 - x^4}.$$

Example 5.4. Differentiation of (5.6) gives

$$(5.8) \quad \int_0^\infty \frac{x e^{-ax} dx}{\beta^3 + \beta^2 x + \beta x^2 + x^3} = \frac{1}{2\beta} \{ \text{ci}(a\beta)(\sin a\beta - \cos(a\beta)) - \text{si}(a\beta)(\sin a\beta + \cos(a\beta)) - e^{a\beta} \text{Ei}(-a\beta) \}$$

This is entry **3.357.2** in [2].

Example 5.5. Differentiating (5.8) produces entry **3.357.3**:

$$(5.9) \quad \int_0^\infty \frac{x^2 e^{-ax} dx}{\beta^3 + \beta^2 x + \beta x^2 + x^3} = \frac{1}{2} \{ -\text{ci}(a\beta)(\sin a\beta + \cos(a\beta)) - \text{si}(a\beta)(\sin a\beta - \cos(a\beta)) - e^{a\beta} \text{Ei}(-a\beta) \}.$$

The identity

$$(5.10) \quad \frac{1}{\beta^3 - \beta^2 x + \beta x^2 - x^3} = \frac{\beta + x}{\beta^4 - x^4}$$

and the method used to establish the last three entries produces proofs of the next three.

Example 5.6. Entry **3.357.4** is

$$(5.11) \quad \int_0^\infty \frac{e^{-ax} dx}{\beta^3 - \beta^2 x + \beta x^2 - x^3} = \frac{1}{2\beta^2} \{ \text{ci}(a\beta)(\sin a\beta - \cos(a\beta)) - \text{si}(a\beta)(\sin a\beta + \cos(a\beta)) + e^{-a\beta} \text{Ei}(a\beta) \}$$

and **3.357.5** is

$$(5.12) \quad \int_0^\infty \frac{x e^{-ax} dx}{\beta^3 - \beta^2 x + \beta x^2 - x^3} = \frac{1}{2\beta} \{ -\text{ci}(a\beta)(\sin a\beta + \cos(a\beta)) - \text{si}(a\beta)(\sin a\beta - \cos(a\beta)) + e^{-a\beta} \text{Ei}(a\beta) \}$$

and, finally, entry **3.357.6** is

$$(5.13) \quad \int_0^\infty \frac{x^2 e^{-ax} dx}{\beta^3 - \beta^2 x + \beta x^2 - x^3} = \frac{1}{2} \{ \text{ci}(a\beta)(\cos a\beta - \sin(a\beta)) + \text{si}(a\beta)(\cos a\beta + \sin(a\beta)) + e^{-a\beta} \text{Ei}(a\beta) \}.$$

6. Entries involving absolute values

This section presents the evaluation of some entries in [2] where the integrand contains variations of the function $\ln|x|$.

Example 6.1. Entry 4.337.3 states that

$$(6.1) \quad \int_0^\infty e^{-\mu x} \ln|a-x| dx = \frac{1}{\mu} [\ln a - e^{-a\mu} \text{Ei}(a\mu)].$$

To establish this entry observe that the singularity at $x = a$ is integrable and that

$$(6.2) \quad \frac{d}{dx} \ln|a-x| = \frac{1}{a-x}.$$

Integration by parts produces

$$\begin{aligned} \int_0^\infty e^{-\mu x} \ln|a-x| dx &= -\frac{1}{\mu} \int_0^\infty \ln|x-a| de^{-\mu x} \\ &= -\frac{1}{\mu} \left(-\log a - e^{-\mu a} \int_0^\infty \frac{e^{-\mu x}}{x-a} dx \right) \\ &= \frac{1}{\mu} \left(\ln a + e^{-\mu t} \int_{-\mu a}^\infty \frac{e^{-u}}{u} du \right) \\ &= \frac{1}{\mu} (\ln a - e^{-\mu a} \text{Ei}(\mu a)). \end{aligned}$$

This is the result.

Example 6.2. Entry 4.337.4 states that

$$(6.3) \quad \int_0^\infty e^{-\mu x} \ln \left| \frac{\beta}{\beta-x} \right| dx = \frac{1}{\mu} e^{-\beta\mu} \text{Ei}(\beta\mu).$$

This evaluation is obtained directly from (6.1) and the identity

$$(6.4) \quad \int_0^\infty e^{-\mu x} \ln \left| \frac{\beta}{\beta-x} \right| dx = \ln|\beta| \int_0^\infty e^{-\mu x} dx - \int_0^\infty e^{-\mu x} \ln|\beta-x| dx.$$

7. Some integrals involving the logarithm function

The exponential integral function Ei allows the evaluation of a variety of entries in [2] containing a logarithmic term. For instance 4.212.1:

$$(7.1) \quad \int_0^1 \frac{dx}{a + \ln x} = e^{-a} \text{Ei}(a)$$

follows from the change of variables $t = a + \ln x$. Similarly, 4.212.2:

$$(7.2) \quad \int_0^1 \frac{dx}{a - \ln x} = -e^a \text{Ei}(-a)$$

is evaluated using $t = a - \ln x$.

We now consider the family

$$(7.3) \quad f_n(a) := \int_0^1 \frac{dx}{(a + \ln x)^n}.$$

The change of variables $t = a + \ln x$ gives

$$(7.4) \quad f_n(a) = e^{-a} \int_{-\infty}^a t^{-n} e^t dt.$$

Integrate by parts to produce

$$(7.5) \quad \int_{-\infty}^a \frac{e^t dt}{t^n} = \frac{e^a a^{1-n}}{1-n} - \frac{1}{1-n} \int_{-\infty}^a \frac{e^t dt}{t^{n-1}}.$$

This yields a recurrence for the integrals $f_n(a)$:

$$(7.6) \quad f_n(a) = -\frac{a^{1-n}}{n-1} + \frac{1}{n-1} f_{n-1}(a).$$

The initial value is given in **4.212.1**. From here we deduce and prove by induction, formula **4.212.8**:

$$(7.7) \quad \int_0^1 \frac{dx}{(a + \ln x)^n} = \frac{e^{-a}}{(n-1)!} \text{Ei}(a) - \frac{1}{(n-1)!} \sum_{k=1}^{n-1} \frac{(n-k-1)!}{a^{n-k}}.$$

Using (7.4) we obtain **3.351.4**:

$$(7.8) \quad \int_a^\infty \frac{e^{-px} dx}{x^{n+1}} = \frac{(-1)^{n+1} p^n}{n!} \text{Ei}(-ap) + \frac{e^{-ap}}{a^n n!} \sum_{k=0}^{n-1} (-1)^k p^k a^k (n-k-1)!$$

The integral **4.212.3**:

$$(7.9) \quad \int_0^1 \frac{dx}{(a + \ln x)^2} = -\frac{1}{a} + e^{-a} \text{Ei}(a)$$

is the special case $n = 2$ of (7.7). The integral **4.212.5**:

$$(7.10) \quad \int_0^1 \frac{\ln x dx}{(a + \ln x)^2} = 1 + (1-a)e^{-a} \text{Ei}(a)$$

can be obtained from

$$(7.11) \quad \frac{\ln x}{(a + \ln x)^2} = \frac{1}{a + \ln x} - \frac{a}{(a + \ln x)^2}.$$

Similar arguments produce **4.212.9**:

$$(7.12) \quad \int_0^1 \frac{dx}{(a + \ln x)^n} = \frac{(-1)^n e^a \text{Ei}(-a)}{(n-1)!} + \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=1}^{n-1} (n-k-1)! (-a)^{k-n}.$$

The formula **4.212.4**:

$$(7.13) \quad \int_0^1 \frac{dx}{(a - \ln x)^2} = \frac{1}{a} + e^a \text{Ei}(-a)$$

is the special case $n = 2$. Writing

$$(7.14) \quad \ln x = a - (a - \ln x)$$

we obtain the evaluation of **4.212.6**:

$$(7.15) \quad \int_0^1 \frac{\ln x \, dx}{(a - \ln x)^2} = 1 + (1 + a)e^a \text{Ei}(-a).$$

8. The exponential scale

Several of the entries in [2] contain integrals that can be reduced to the definition of the exponential integral. This section contains some of them.

Example 8.1. Entry **4.331.2** states that

$$(8.1) \quad \int_1^\infty e^{-\mu x} \ln x \, dx = -\frac{1}{\mu} \text{Ei}(-\mu), \text{ for } \text{Re } \mu > 0.$$

To evaluate this entry, assume $\mu > 0$ and integrate by parts to obtain

$$(8.2) \quad \int_1^\infty e^{-\mu x} \ln x \, dx = \frac{1}{\mu} \int_1^\infty \frac{e^{-\mu x}}{x} \, dx.$$

The change of variables $s = -\mu x$ now gives the result for $\mu \in \mathbb{R}$. The case $\mu \in \mathbb{C}$ follows by analytic continuation.

Example 8.2. Entry **4.337.1**

$$(8.3) \quad \int_0^\infty e^{-\mu x} \ln(\beta + x) \, dx = \frac{1}{\mu} [\ln \beta - e^{\mu\beta} \text{Ei}(-\beta\mu)], \text{ for } |\arg \beta| < \pi, \text{Re } \mu > 0$$

can be transformed to **4.331.2** by simple changes of variables. Start with $\beta > 0$ and make the change of variables $x = \beta t$ to obtain

$$(8.4) \quad \int_0^\infty e^{-\mu x} \ln(\beta + x) \, dx = \frac{\ln \beta}{\mu} + \beta \int_0^\infty e^{-\mu\beta t} \ln(1 + t) \, dt.$$

The change of variables $s = t + 1$ and Entry **4.331.2** gives the result.

Example 8.3. Entry **4.337.2** is

$$(8.5) \quad \int_0^\infty e^{-\mu x} \ln(1 + \beta x) \, dx = -\frac{1}{\mu} e^{\mu/\beta} \text{Ei}(-\mu/\beta).$$

The change of variables $t = \beta x$ reduces this integral to **4.337.1** with $\mu \mapsto \mu/\beta$ and $\beta \mapsto 1$.

The change of variables $t = -ae^{nu}$ produces

$$(8.6) \quad \text{Ei}(x) = -n \int_c^\infty \exp(-ae^{nu}) \, du,$$

where $c = \frac{1}{n} \ln(-x/a)$. The choice $x = -a$ produces

$$(8.7) \quad \text{Ei}(-a) = -n \int_0^\infty \exp(-ae^{nu}) \, du.$$

This appears as **3.327** in [2].

Some further examples of entries in [2], containing the exponential integral function, will be described in a future publication.

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