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The integrals in Gradshteyn and Ryzhik. Part 27: More logarithmic examples

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Abstract. The table of Gradshteyn and Ryzhik contains many entries where the integrand is a combination of an elementary function and the logarithmic of another function of the same type. This paper presents proofs of some of these. A sample of examples where the elementary function is replaced by an algebraic function is also discussed.

1. Introduction

The compendium [**5**] contains a large collection of evaluation of integrals of the form

$$\int_a^b R_1(x) \ln R_2(x)\, dx \tag{1.1}$$

where R_1 and R_2 are rational functions. The first paper in this series [**9**] considered the family

$$f_n(a) = \int_0^\infty \frac{\ln^{n-1} x\, dx}{(x-1)(x+a)}, \text{ for } n \geqslant 2 \text{ and } a > 0. \tag{1.2}$$

The function $f_n(a)$ is given explicitly by

$$\begin{aligned} f_n(a) &= \frac{(-1)^n (n-1)!}{1+a} \left[1 + (-1)^n \right] \zeta(n) \\ &+ \frac{1}{n(1+a)} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} (2^{2j}-2)(-1)^{j-1} B_{2j} \pi^{2j} (\log a)^{n-2j}. \end{aligned} \tag{1.3}$$

Here $\zeta(s)$ is the Riemann zeta function and B_{2j} is the Bernoulli number. In particular, (1.3) shows that $(1+a)f_n(a)$ is a polynomial in $\log a$.

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Other papers in this series [**3, 8, 10**] and also [**6**] considered examples of integrals of this type. The results in [**3**] can be used to provide explicit expressions for an integral of the type considered here, when the poles of the rational function R_2 in (1.1) have real or purely imaginary parts. The present paper is a continuation of this work.

2. Some examples involving rational functions

This section considers of integrals of the form

$$\int_a^b R_1(x) \ln R_2(x)\, dx \tag{2.1}$$

where R_1 and R_2 are rational functions.

Example 2.1. Entry **4.234.4** is

$$\int_0^\infty \frac{1-x^2}{(1+x^2)^2} \ln x\, dx = -\frac{\pi}{2} \tag{2.2}$$

To evaluate this entry, observe that

$$\frac{d}{dx} \frac{x}{1+x^2} = \frac{1-x^2}{(1+x^2)^2}, \tag{2.3}$$

and integrating by parts gives

$$\int_0^\infty \frac{1-x^2}{(1+x^2)^2} \ln x\, dx = -\int_0^\infty \frac{dx}{1+x^2} = -\frac{\pi}{2}. \tag{2.4}$$

Example 2.2. Entry **4.234.5** states that

$$\int_0^1 \frac{x^2 \ln x\, dx}{(1-x^2)(1+x^4)} = -\frac{\pi^2}{16(2+\sqrt{2})}. \tag{2.5}$$

To prove this use the method of partial fraction to obtain

$$\int_0^1 \frac{x^2 \ln x\, dx}{(1-x^2)(1+x^4)} = \frac{1}{4}\int_0^1 \frac{\ln x\, dx}{1-x} + \frac{1}{4}\int_0^1 \frac{\ln x\, dx}{1+x} + \frac{1}{2}\int_0^1 \frac{(x^2-1)\ln x\, dx}{1+x^4}. \tag{2.6}$$

The first integral is $-\pi^2/6$ according to entry **4.231.2** and the second one is $-\pi^2/12$ from entry **4.231.1**. These entries were established in [**1**]. This gives

$$\int_0^1 \frac{x^2 \ln x\, dx}{(1-x^2)(1+x^4)} = -\frac{\pi^2}{16} + \frac{1}{2}\int_0^1 \frac{(x^2-1)\ln x\, dx}{1+x^4}. \tag{2.7}$$

To evaluate the last integral, observe that

$$\frac{x^2-1}{1+x^4} = \sum_{n=0}^\infty (-1)^{n-1} x^{4n} + \sum_{n=0}^\infty (-1)^n x^{4n+2}. \tag{2.8}$$

Now recall the *digamma function* $\psi(z) = \Gamma'(z)/\Gamma(z)$ and the expansion of its derivative

$$\psi'(x) = \sum_{n=0}^\infty \frac{1}{(x+n)^2}. \tag{2.9}$$

Details about this function may be found in [**4**] and [**13**]. This gives

$$\int_0^1 \frac{(x^2-1)\ln x\,dx}{1+x^4} = \frac{1}{64}\left[\psi'\left(\frac{1}{8}\right) - \psi'\left(\frac{3}{8}\right) - \psi'\left(\frac{5}{8}\right) + \psi'\left(\frac{7}{8}\right)\right]. \tag{2.10}$$

The classical relation

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \tag{2.11}$$

can be shifted to produce

$$\Gamma\left(\tfrac{1}{2}+x\right)\Gamma\left(\tfrac{1}{2}-x\right) = \frac{\pi}{\cos \pi x}. \tag{2.12}$$

Logarithmic differentiation shows that the digamma function satisfies

$$\psi\left(\frac{1}{2}+x\right) - \psi\left(\frac{1}{2}-x\right) = \pi \tan \pi x. \tag{2.13}$$

This appears as Entry **8.365.9** in [**5**]. Differentiation produces

$$\psi'\left(\frac{1}{2}+x\right) + \psi'\left(\frac{1}{2}-x\right) = \pi^2 \sec^2 \pi x. \tag{2.14}$$

Now use (2.14) and group 1/8 with 7/8 and 3/8 with 5/8 to produce

$$\int_0^1 \frac{(x^2-1)\,\ln x\,dx}{1+x^4} = \frac{1}{64}\left(\frac{4\pi^2}{2-\sqrt{2}} - \frac{4\pi^2}{2+\sqrt{2}}\right) = \frac{\pi^2}{8\sqrt{2}}. \tag{2.15}$$

Note 2.3. The reader should evaluate the family of integrals

$$I_n = \int_0^1 \frac{x^{2n}\,\ln x}{(1-x^2)(1+x^4)^n}\,dx, \quad n \in \mathbb{N}, \tag{2.16}$$

by the method described here. The computation of the first few special values indicates an interesting arithmetic structure of the answer.

3. An entry involving the Poisson kernel for the disk

The section discusses a single entry in [**5**], where the integrand involves the Poisson kernel for the disk. Further examples of this type will be presented in a future publication.

Example 3.1. The next evaluation is Entry **4.233.5**:

$$\int_0^\infty \frac{\ln x\,dx}{x^2+2xa\cos t+a^2} = \frac{t}{\sin t}\frac{\ln a}{a}. \tag{3.1}$$

The integrand is related to the *Poisson kernel* for the unit disk $D = \{z \in \mathbb{C} : \, |z| < 1\}$.

Theorem 3.2. Define

$$\mathcal{P}_r(\theta) = \text{Re}\,\frac{1+re^{i\theta}}{1-re^{i\theta}} \tag{3.2}$$

then $\mathcal{P}_r(\theta)$ is given by

$$\mathcal{P}_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta} = \frac{1-r^2}{1-2r\cos\theta+r^2}. \tag{3.3}$$

Moreover, given f defined on the boundary of D, the expression

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{P}_r(\theta - t) f(e^{it})\, dt \tag{3.4}$$

for $0 \leqslant r < 1$, is a harmonic function on D and it has a radial limit which agrees with f almost everywhere on the boundary of D.

The form of the Poisson kernel can be used to establish the next result.

Lemma 3.3. For a, $x \in \mathbb{R}$ with $|x| < |a|$,

$$\sum_{k=0}^{\infty} \frac{(-1)^k \sin((k+1)t) x^k}{a^k} = \frac{a^2 \sin t}{x^2 + 2ax\cos t + a^2}. \tag{3.5}$$

Note 3.4. The Chebyshev polynomial of the second kind $U_n(t)$ is defined by the identity

$$\frac{\sin((n+1)\theta)}{\sin\theta} = U_n(\cos\theta). \tag{3.6}$$

The result of Lemma 3.3 can be written as

$$\sum_{k=0}^{\infty} U_k(t) x^k = \frac{1}{x^2 - 2x\cos t + 1}. \tag{3.7}$$

Lemma 3.3 produces

$$\int_0^R \frac{x^s\, dx}{x^2 + 2ax\cos t + a^2} = \frac{1}{a^2 \sin t} \sum_{k=0}^{\infty} \frac{(-1)^k \sin((k+1)t) R^{k+s+1}}{a^k\, (k+s+1)}. \tag{3.8}$$

Now write $\sin((k+1)t)$ in terms of exponential to obtain an expression for the previous integral as

$$\int_0^R \frac{x^s\, dx}{x^2 + 2ax\cos t + a^2} = \frac{R^{s+1}}{2ia^2 \sin t}\left(e^{it}\Phi\left(-\frac{R}{ae^{it}}, 1, s+1\right) - e^{-it}\Phi\left(-\frac{R}{ae^{-it}}, 1, s+1\right)\right)$$

where

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(a+k)^s} \tag{3.9}$$

is the *Lerch Phi function*.

Now differentiate with respect to s and let $s \to 0$ to produce

$$\begin{aligned} \int_0^R \frac{\ln x\, dx}{x^2 + 2ax\cos t + a^2} &= \frac{i \ln R}{2a\sin t}\left(\log(1 + e^{-it}R/a) - \log(1 + e^{it}R/a)\right) \\ &+ \frac{i}{2a\sin t}\left(\mathrm{Li}_2(-e^{-it}R/a) - \mathrm{Li}_2(-e^{-it}R/a)\right), \end{aligned} \tag{3.10}$$

where

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \tag{3.11}$$

is the *dilogarithm* function. Then use the identity

$$i\left(\mathrm{Li}_2(-e^{-it}R/a) - \mathrm{Li}_2(-e^{-it}R/a)\right) = -\int_0^t \ln\left(\frac{a^2+2Ra\cos z+R^2}{a^2}\right)\,dz \tag{3.12}$$

to obtain

$$\begin{aligned}\int_0^R \frac{\ln x\,dx}{x^2+2ax\cos t+a^2} &= \frac{i\ln R}{2a\sin t}\left(\log(1+e^{-it}R/a)-\log(1+e^{it}R/a)\right)\\ &\quad -\frac{1}{2a\sin t}\int_0^t \ln\left(\frac{a^2+2Ra\cos z+R^2}{a^2}\right)\,dz.\end{aligned} \tag{3.13}$$

The next step is to differentiate (3.13) with respect to t and let $R\to\infty$. The left-hand side produces

$$T_1(a,t) = \int_0^\infty \frac{2ax\ln x\sin t\,dx}{(x^2+2ax\cos t+a^2)^2}. \tag{3.14}$$

Direct differentiation of the right-hand side yields

$$T_2(a,t) = \lim_{R\to\infty} V_1(R;a,t)+V_2(R;a,t) \tag{3.15}$$

where

$$V_1(R;a,t) = \frac{R\ln R(R+a\cos t)}{a\sin t(a^2+2aR\cos t+R^2)} - \frac{1}{2a\sin t}\ln\left(\frac{a^2+2aR\cos t+R^2}{a^2}\right) \tag{3.16}$$

and

$$\begin{aligned}V_2(R;a,t) &= \frac{i\cos t\ln R}{2a\sin^2 t}\left(\log(1+e^{it}R/a)-\log(1+e^{-it}R/a)\right)\\ &\quad +\frac{\cos t}{2a\sin^2 t}\int_0^t \ln\left(\frac{a^2+2Ra\cos z+R^2}{a^2}\right)\,dz.\end{aligned} \tag{3.17}$$

Proposition 3.5. The function $T_2(a,t)$ is given

$$T_2(a,t) = -\frac{\ln a}{2a\sin t}\left(t\cot t-1\right). \tag{3.18}$$

Proof. Start with the computation of the limiting behavior of $V_1(R;a,t)$. The claim that

$$\lim_{R\to\infty} V_1(R;a,t) = \frac{\ln a}{a\sin(t)}. \tag{3.19}$$

is verified first.

First note that since

$$\lim_{R\to\infty}\frac{R\ln R}{a^2+2aR\cos(t)+R^2} = 0, \tag{3.20}$$

then

$$\lim_{R\to\infty} V_1(R;a,t) = \frac{1}{a\sin t}\lim_{R\to\infty}\left(\frac{R^2\ln R}{a^2+2aR\cos t+R^2} - \frac{1}{2}\ln(a^2+2aR\cos t+R^2)+\ln a\right).$$

The claim is equivalent to

$$\lim_{R\to\infty}\left(\frac{R^2\ln R}{a^2+2aR\cos t+R^2}-\frac{1}{2}\ln(a^2+2aR\cos t+R^2)\right)=0. \tag{3.21}$$

The identities

$$\frac{R^2\ln R}{a^2+2aR\cos t+R^2}=\frac{\ln R}{a^2/R^2+2a\cos t/R+1} \tag{3.22}$$

and

$$\frac{1}{2}\ln(a^2+2aR\cos t+R^2)=\ln R+\frac{1}{2}\ln(a^2/R^2+2a\cos t/R+1) \tag{3.23}$$

can be used to see that the left-hand side of (3.21) is equivalent to

$$\lim_{R\to\infty}\left(\ln R\left(\frac{1}{a^2/R^2+2a\cos t/R+1}-1\right)-\frac{1}{2}\ln(a^2/R^2+2a\cos t/R+1)\right)=0.$$

It is clear that the second term vanishes as $R\to\infty$. For the first term, observe that

$$\frac{1}{a^2/R^2+2a\cos(t)/R+1}-1=-\frac{2a\cos t}{R}+O\left(\frac{1}{R^2}\right) \tag{3.24}$$

and thus the first term also vanishes as $R\to\infty$. This concludes the proof.

The next step is to verify that

$$\begin{aligned} V_2(R;a,t) \quad = \quad & \frac{i\cot t\ln R}{2a\sin^2 t}\left(\log(1+e^{it}R/a)-\log(1+e^{-it}R/a)\right) \\ & +\frac{\cos t}{2a\sin^2 t}\int_0^t\ln\left(\frac{a^2+2aR\cos z+R^2}{a^2}\right)dz \end{aligned} \tag{3.25}$$

satisfies

$$\lim_{R\to\infty}V_2(R;a,t)=-\frac{t\cos t}{a\sin^2 t}\ln a. \tag{3.26}$$

The proof begins with the identity

$$\log(1+b/x)=\log(b/x)+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{nb^n} \tag{3.27}$$

to obtain

$$\log(1+e^{it}R/a)-\log(1+e^{-it}R/a)=\log(e^{it})-\log(e^{-it})+O(a/R),\ \text{as } R\to\infty. \tag{3.28}$$

The bounds $0 < t < \pi$ imply $\log(e^{it}) - \log(e^{-it}) = 2it$. This gives

$$\begin{aligned}
\lim_{R\to\infty} V_2(R;a,t) &= \lim_{R\to\infty} \left(\frac{\cos t}{2a\sin^2 t} \int_0^t \ln\left(\frac{a^2+2aR\cos z+R^2}{a^2}\right) dz - \frac{t\cos z \ln R}{a\sin^2 t}\right) \\
&= \lim_{R\to\infty} \frac{\cos t}{2a\sin^2 t} \left(\int_0^t \ln\left(\frac{a^2+2aR\cos z+R^2}{a^2}\right) dz - 2t\ln R\right) \\
&= \lim_{R\to\infty} \frac{\cos t}{2a\sin^2 t} \left(\int_0^t \ln\left(\frac{a^2+2aR\cos z+R^2}{a^2}\right) - \ln(R^2)\, dz\right) \\
&= \lim_{R\to\infty} \frac{\cos t}{2a\sin^2 t} \left(\int_0^t \left[\ln\left(a^2+2aR\cos z+R^2\right) - \ln(R^2)\right] dz - 2t\ln a\right).
\end{aligned}$$

The identity

$$\int_0^t \left[\ln\left(a^2+2aR\cos z+R^2\right) - \ln(R^2)\right] dz = \int_0^t \ln\left(\frac{a^2}{R^2} + \frac{2a\cos z}{R} + 1\right) dz$$

gives the result. The proof of the Proposition is finished. $\square$

The evaluation of entry **4.233.5** is now obtained from the identity $T_1(a,t) = T_2(a,t)$. Observe that this implies

$$\int_0^\infty \frac{2ax\ln x \sin t\, dx}{(x^2+2ax\cos t+a^2)^2} = -\frac{\ln a}{a\sin t}\left(t\cot t - 1\right). \tag{3.29}$$

Integrating with respect to t gives (3.1). Entry **4.231.8** in [**5**], established in [**3**],

$$\int_0^\infty \frac{\ln x\, dx}{x^2+a^2} = \frac{\pi \ln a}{2a} \tag{3.30}$$

can be used to show that the implicit constant of integration actually vanishes. The evaluation is complete.

4. Some rational integrands with a pole at $x = 1$

This section contains proofs of the four entries appearing in Section 4.235. These are integrals of the form

$$f(a,b,c) := \int_0^\infty \frac{x^b - x^c}{1-x^a} \ln x\, dx \tag{4.1}$$

where a, b, $c \in \mathbb{N}$. These integrals are evaluated using entry **4.254.2**

$$\int_0^\infty \frac{x^{p-1}\ln x}{1-x^q}\, dx = -\frac{\pi^2}{q^2\sin^2\frac{\pi p}{q}}. \tag{4.2}$$

To obtain this formula, start from **3.231.6**

$$\int_0^\infty \frac{x^{p-1}-x^{q-1}}{1-x}\, dx = \pi\left(\cot \pi p - \cot \pi q\right), \tag{4.3}$$

established in [**7**] and make the change of variables $t = x^q$ to produce

$$\begin{aligned} \int_0^\infty \frac{x^{p-1}-1}{1-x^q}\,dx &= -\frac{1}{q}\int_0^\infty \frac{t^{1/q-1}-t^{p/q-1}}{1-t}\,dt \\ &= -\frac{\pi}{q}\left(\cot\frac{\pi}{q} - \cot\frac{\pi p}{q}\right). \end{aligned}$$

Differentiating with respect to p gives (4.2).

LEMMA 4.1. *Let a, b, $c \in \mathbb{R}$. Then*

$$\int_0^\infty \frac{x^{b-1}-x^{c-1}}{1-x^a}\ln x\,dx = -\frac{\pi^2}{a^2}\frac{\sin(c_1-b_1)\sin(c_1+b_1)}{\sin^2 b_1\,\sin^2 c_1} \tag{4.4}$$

where $b_1 = \pi b/a$ and $c_1 = \pi c/a$.

PROOF. Simply write

$$\int_0^\infty \frac{x^{b-1}-x^{c-1}}{1-x^a}\ln x\,dx = \int_0^\infty \frac{x^{b-1}}{1-x^a}\ln x\,dx - \int_0^\infty \frac{x^{c-1}}{1-x^a}\ln x\,dx$$

and use (4.2). □

The four entries in Section 4.235 are established next.

Example 4.1. Entry **4.235.1** states that

$$\int_0^\infty \frac{(1-x)x^{n-2}}{1-x^{2n}}\ln x\,dx = -\frac{\pi^2}{4n^2}\tan^2\frac{\pi}{2n}. \tag{4.5}$$

Lemma 4.1 is used with $a = 2n$, $b = n-1$ and $c = n$. This gives

$$b_1 = \frac{\pi}{2} - \frac{\pi}{2n} \text{ and } c_1 = \frac{\pi}{2}. \tag{4.6}$$

and

$$\int_0^\infty \frac{(1-x)x^{n-2}}{1-x^{2n}}\ln x\,dx = -\frac{\pi^2}{4n^2}\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)\sin\left(\frac{\pi}{2}+\frac{\pi}{2n}\right)}{\sin^2\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)} = -\frac{\pi^2}{4n^2}\tan^2\frac{\pi}{2n}.$$

Example 4.2. Entry **4.235.2** is

$$\int_0^\infty \frac{(1-x^2)x^{m-1}}{1-x^{2n}}\ln x\,dx = -\frac{\pi^2}{4n^2}\frac{\sin\left(\frac{m+1}{n}\pi\right)\sin\left(\frac{\pi}{n}\right)}{\sin^2\left(\frac{\pi m}{2n}\right)\sin^2\left(\frac{(m+2)}{2n}\pi\right)}. \tag{4.7}$$

Lemma 4.1 is now used with $a = 2n$, $b = m$ and $c = m+2$. This gives

$$c_1 - b_1 = \frac{\pi}{n} \text{ and } c_1 + b_1 = \frac{\pi}{n}(m+1) \tag{4.8}$$

to produce the result.

Example 4.3. Entry **4.235.3** states that

$$\int_0^\infty \frac{(1-x^2)x^{n-3}}{1-x^{2n}}\ln x\,dx = -\frac{\pi^2}{4n^2}\tan^2\frac{\pi}{n}. \tag{4.9}$$

The values $a = 2n$, $b = n-2$ and $c = n$ give

$$b_1 = \frac{\pi}{2} - \frac{\pi}{n} \text{ and } c_1 = \frac{\pi}{2}. \tag{4.10}$$

This verifies the claim.

Example 4.4. Entry **4.235.4** appears as

$$\int_0^1 \frac{x^{m-1}+x^{n-m-1}}{1-x^n} \ln x \, dx = -\frac{\pi^2}{n^2 \sin^2 \frac{\pi m}{n}}. \tag{4.11}$$

The change of variables $t = 1/x$ shows that the integral over $[1, \infty)$ is equal to that over $[0, 1]$, therefore this entry should be written as

$$\int_0^\infty \frac{x^{m-1}+x^{n-m-1}}{1-x^n} \ln x \, dx = -\frac{2\pi^2}{n^2 \sin^2 \frac{\pi m}{n}}, \tag{4.12}$$

to be consistent with the other entries in this section. The proof comes from Lemma 4.1 with $a = n$, $b = m$ and $c = n - m$.

5. Some singular integrals

The table [**5**] contains a variety of singular integrals of the form being discussed here. The examples considered in this section are evaluated employing the formula

$$\int_0^\infty \frac{t^{\mu-1}\, dt}{1-t} = \pi \cot \pi\mu. \tag{5.1}$$

To verify this evaluation, transform the integral over $[1, \infty)$ to $[0, 1]$ by the change of variables $x \mapsto 1/x$. This gives

$$\int_0^\infty \frac{t^{\mu-1}\, dt}{1-t} = \int_0^1 \frac{t^{\mu-1}-t^{-\mu}}{1-t}\, dt. \tag{5.2}$$

This is entry **3.231.1**. It was established in [**7**].

Differentiating with respect to μ, the formula (5.1) gives

$$\int_0^\infty \frac{t^{\mu-1} \ln t}{1-t}\, dt = -\frac{\pi^2}{\sin^2 \pi\mu}, \tag{5.3}$$

and the change of variables $t = x^a$ gives

$$\omega(a,b) := \int_0^\infty \frac{x^{b-1} \ln x}{1-x^a}\, dx = -\frac{\pi^2}{a^2 \sin^2\left(\frac{\pi b}{a}\right)}. \tag{5.4}$$

Example 5.1. Entry **4.251.2** states that

$$\int_0^\infty \frac{x^{\mu-1} \ln x}{a-x} = \pi a^{\mu-1} \left(\ln a \, \cot(\pi\mu) - \frac{\pi}{\sin^2 \pi\mu} \right). \tag{5.5}$$

The change of variables $x = at$ yields

$$\int_0^\infty \frac{x^{\mu-1} \ln x}{a-x} = a^{\mu-1} \int_0^\infty \frac{t^{\mu-1} \ln t}{1-t}\, dt + a^{\mu-1} \ln a \int_0^\infty \frac{t^{\mu-1}\, dt}{1-t}. \tag{5.6}$$

The result now follows from (5.1) and (5.3). It is probably clearer to write this entry as

$$\int_0^\infty \frac{x^{\mu-1} \ln x}{a-x} = \pi a^{\mu-1} \left(\frac{\ln a}{\tan \pi\mu} - \frac{\pi}{\sin^2 \pi\mu} \right), \tag{5.7}$$

to avoid possible confusions.

Example 5.2. Entry **4.252.3** is

$$\int_0^\infty \frac{x^{p-1}\,\ln x}{1-x^2}\,dx = -\frac{\pi^2}{4}\mathrm{cosec}^2\,\frac{\pi p}{2}. \tag{5.8}$$

This is $\omega(2,p)$ and the result follows from (5.4).

Example 5.3. Entry **4.255.3** states that

$$\int_0^\infty \frac{1-x^p}{1-x^2}\,\ln x\,dx = \frac{\pi^2}{4}\tan^2\left(\frac{\pi p}{2}\right). \tag{5.9}$$

This is $\omega(1,2)-\omega(p+1,2)$ and the result comes from (5.4).

Example 5.4. Entry **4.252.1** is written as

$$\int_0^\infty \frac{x^{\mu-1}\ln x\,dx}{(x+a)(x+b)} = \frac{\pi}{(b-a)\sin\pi\mu}\left[a^{\mu-1}\ln a - b^{\mu-1}\ln b - \pi\frac{a^{\mu-1}-b^{\mu-1}}{\tan\pi\mu}\right].$$

This value follows from the partial fraction decomposition

$$\frac{1}{(x+a)(x+b)} = \frac{1}{b-a}\frac{1}{x+a} - \frac{1}{b-a}\frac{1}{x+b} \tag{5.10}$$

and entry **4.251.1**

$$\int_0^\infty \frac{x^{\mu-1}\,\ln x}{x+c}\,dx = \frac{\pi c^{\mu-1}}{\sin\pi\mu}\left(\ln c - \pi\cot\pi\mu\right), \tag{5.11}$$

established in [**11**]. Differentiating (5.11) with respect to c yields

$$\int_0^\infty \frac{x^{\mu-1}\,\ln x}{(x+c)^2}\,dx = -\frac{(\mu-1)c^{\mu-2}\pi}{\sin\pi\mu}\left(\ln c - \pi\cot\pi\mu + \frac{1}{\mu-1}\right). \tag{5.12}$$

This is entry **4.252.4**.

Example 5.5. Entry **4.257.1**

$$\int_0^\infty \frac{x^\mu\,\ln(x/a)\,dx}{(x+a)(x+b)} = \frac{\pi\left[b^\mu\ln(b/a) + \pi(a^\mu - b^\mu)\cot\pi\mu\right]}{(b-a)\sin\pi\mu} \tag{5.13}$$

follows from (5.11) and the beta integral

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{x+a} = \frac{\pi a^{\mu-1}}{\sin\pi\mu}. \tag{5.14}$$

This appears as entry **3.194.3** and it was established in [**11**].

Example 5.6. The change of variables $t=x^q$ gives

$$\int_0^\infty \frac{x^{p-1}\,dx}{1-x^q} = \frac{1}{q}\int_0^\infty \frac{t^{p/q-1}\,dx}{1-t} = \frac{\pi}{q}\cot\left(\frac{\pi p}{q}\right) \tag{5.15}$$

from (5.3). This is entry **3.241.3**. The special case $q=1$ gives

$$\int_0^\infty \frac{x^{p-1}\,dx}{1-x} = \pi\cot\pi p. \tag{5.16}$$

Differentiating with respect to p produces

$$\int_0^\infty \frac{x^{p-1}\,\ln x}{1-x}\,dx = -\frac{\pi^2}{\sin^2 \pi p}. \tag{5.17}$$

The partial fraction decomposition

$$\frac{1}{(x+a)(x-1)} = \frac{1}{a+1}\frac{1}{x-1} - \frac{1}{a+1}\frac{1}{x+a} \tag{5.18}$$

then produces entry **4.252.2**

$$\int_0^\infty \frac{x^{\mu-1}\,\ln x}{(x+a)(x-1)}\,dx = \frac{\pi}{(a+1)\sin^2 \pi\mu}\left[\pi - a^{\mu-1}\left(\ln a\, \sin \pi\mu - \pi \cos \pi\mu\right)\right]. \tag{5.19}$$

Example 5.7. The change of variables $t = x^q$ produces

$$\int_0^\infty \frac{\ln x\,dx}{x^p(x^q-1)} = -\frac{1}{q^2}\int_0^\infty \frac{t^{(1-p)/q-1}\,\ln t\,dt}{1-t}. \tag{5.20}$$

Then, (5.3) gives

$$\int_0^\infty \frac{\ln x\,dx}{x^p(x^q-1)} = \frac{\pi^2}{q^2}\frac{1}{\sin^2\left(\frac{p-1}{q}\pi\right)}. \tag{5.21}$$

This is entry **4.254.3**.

Example 5.8. Entry **4.255.2** is

$$\int_0^1 \frac{(1+x^2)x^{p-2}}{1-x^{2p}}\,\ln x\,dx = -\left(\frac{\pi}{2p}\right)^2 \sec^2 \frac{\pi}{2p}. \tag{5.22}$$

The evaluation of this entry starts with entry **3.231.5**

$$\int_0^1 \frac{x^{\mu-1}-x^{\nu-1}}{1-x}\,dx = -\psi(\mu)+\psi(\nu) \tag{5.23}$$

that was establsihed in [**7**]. The special case $\mu = 1$

$$\int_0^1 \frac{1-x^{\nu-1}}{1-x}\,dx = -\psi(1)+\psi(\nu) \tag{5.24}$$

is differentiated with respect to ν to produce

$$\int_0^1 \frac{x^{\nu-1}\,\ln x}{1-x}\,dx = -\psi'(\nu). \tag{5.25}$$

The change of variables $x = t^b$ gives

$$\int_0^1 \frac{t^{c-1}\,\ln t}{1-t^b}\,dt = -\frac{1}{b^2}\psi'\left(\frac{c}{b}\right). \tag{5.26}$$

Therefore

$$\begin{aligned}\int_0^1 \frac{(1-x^2)x^{p-2}}{1-x^{2p}}\,\ln x\,dx &= \int_0^1 \frac{x^{p-2}}{1-x^{2p}}\,\ln x\,dx + \int_0^1 \frac{x^p}{1-x^{2p}}\,\ln x\,dx \\ &= -\frac{1}{4p^2}\left[\psi'\left(\frac{1}{2}-\frac{1}{2p}\right)+\psi'\left(\frac{1}{2}+\frac{1}{2p}\right)\right].\end{aligned}$$

The result now follows from the reflection formula for the polygamma function ψ' given in (2.14).

6. Combinations of logarithms and algebraic functions

This section presents the evaluation of some entries in [**5**] of the form

$$\int_a^b E_1(x) \ln E_2(x)\, dx \tag{6.1}$$

where E_1 or E_2 is an algebraic function. Some of these have appeared in previous papers in this series. For example, entry **4.241.11**

$$\int_0^1 \frac{\ln x\, dx}{\sqrt{x(1-x^2)}} = -\frac{\sqrt{2\pi}}{8}\Gamma^2\left(\frac{1}{4}\right) \tag{6.2}$$

and entry **4.241.5**

$$\int_0^1 \ln x \sqrt{(1-x^2)^{2n-1}}\, dx = -\frac{(2n-1)!!}{4(2n)!!}\pi\left[\psi(n+1)+\gamma+\ln 4\right] \tag{6.3}$$

were evaluated in [**7**]. Here $\psi(x)$ is the digamma function and γ is Euler's constant.

Note 6.1. Define the family of integrals

$$f_n(a) := \int_0^1 \frac{x^a \ln^n x\, dx}{\sqrt{1-x^2}}. \tag{6.4}$$

Special cases include entry **4.241.7**

$$\int_0^1 \frac{\ln x\, dx}{\sqrt{1-x^2}} = -\frac{\pi}{2}\ln 2 \tag{6.5}$$

that was evaluated in [**7**] and entry **4.261.9**

$$\int_0^1 \frac{\ln^2 x\, dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\left(\ln^2 2 + \frac{\pi^2}{12}\right). \tag{6.6}$$

A trigonometric form of the family is obtained by the change of variables $x = \sin t$:

$$f_n(a) = \int_0^{\pi/2} \sin^a t\, \ln^n \sin t\, dt. \tag{6.7}$$

Theorem 6.2. The integral $f_n(a)$ is given by

$$f_n(a) = \lim_{s\to a}\left(\frac{d}{ds}\right)^n h(s), \tag{6.8}$$

where

$$h(s) = \int_0^{\pi/2} \sin^s t\, dt = \frac{1}{2}B\left(\frac{s+1}{2}, \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)}. \tag{6.9}$$

This appears as entry **3.621.5**. Therefore, the evaluation of $f_n(a)$ requires the values of $\Gamma^{(k)}(x)$ for $0 \leqslant k \leqslant n$ at $x = (a+1)/2$ and $x = a/2+1$.

Example 6.3. For example,

$$\begin{aligned} f_1(0) = \int_0^1 \frac{\ln x\, dx}{\sqrt{1-x^2}} &= \lim_{s\to 0} \frac{d}{ds}\left[\frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} \right] \\ &= \frac{\sqrt{\pi}}{4} \frac{\Gamma'(1/2)\Gamma(1) - \Gamma'(1)\Gamma(1/2)}{\Gamma^2(1)}. \end{aligned}$$

The values

$$\Gamma'\left(\tfrac{1}{2}\right) = -\sqrt{\pi}\left(\gamma + 2\ln 2\right),\ \Gamma'(1) = -\gamma,\ \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \text{ and } \Gamma(1) = 1 \tag{6.10}$$

give

$$f_1(0) = -\frac{\pi}{2}\ln 2. \tag{6.11}$$

Proposition 6.4. The derivatives of the gamma function satisfy the recurrence

$$\Gamma^{(n+1)}(x) = \sum_{k=0}^{n} \binom{n}{k} \Gamma^{(k)}(x)\psi^{(n-k)}(x). \tag{6.12}$$

Example 6.5. A direct application of formula (6.8) evaluates entry **4.261.9**

$$f_2(0) = \int_0^1 \frac{\ln^2 x\, dx}{\sqrt{1-x^2}}. \tag{6.13}$$

Indeed, using $\Gamma(1) = 1$, gives

$$f_2(0) = \frac{\sqrt{\pi}}{2}\left[-\tfrac{1}{2}\Gamma'\left(\tfrac{1}{2}\right)\Gamma'(1) + \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right)\Gamma'(1)^2 + \tfrac{1}{4}\Gamma''\left(\tfrac{1}{2}\right) - \tfrac{1}{4}\Gamma\left(\tfrac{1}{2}\right)\Gamma''(1) \right]. \tag{6.14}$$

The values

$$\Gamma''(1) = \gamma^2 + \frac{\pi^2}{6} \text{ and } \Gamma''\left(\tfrac{1}{2}\right) = \frac{1}{2}\pi^{5/2} + \sqrt{\pi}(\gamma + 2\ln 2)^2 \tag{6.15}$$

give the identity (6.6).

It remains to explain the values given in (6.10) and (6.15). The recurrence (6.12) reduces the computation of the derivatives of $\Gamma(x)$ to those of $\psi(x)$. The special values given above come from the next result.

Lemma 6.6. The digamma function satisfies

$$\begin{aligned} \psi^{(n)}(1) &= (-1)^{n+1} n!\, \zeta(n+1) \\ \psi^{(n)}\left(\tfrac{1}{2}\right) &= (-1)^{n+1} n!\, (2^{n+1}-1)\zeta(n+1). \end{aligned}$$

PROOF. This comes directly from (2.9). □

Example 6.7. The values given in Lemma 6.6 yield

$$\begin{aligned} f_3(0) = \int_0^1 \frac{\ln^3 x\, dx}{\sqrt{1-x^2}} &= -\frac{\pi}{8}\left(\pi^2 \ln 2 + 4\ln^3 2 + 6\zeta(3)\right) \\ f_4(0) = \int_0^1 \frac{\ln^4 x\, dx}{\sqrt{1-x^2}} &= \frac{\pi}{480}\left(19\pi^4 + 120\pi^2 \ln^2 2 + 240\ln^4 2 + 1440 \ln 2\, \zeta(3)\right) \end{aligned}$$

and

$$f_1\left(\tfrac{1}{2}\right) = \int_0^1 \frac{\sqrt{x}\,\ln x\,dx}{\sqrt{1-x^2}} = \frac{(\pi-4)}{\sqrt{2\pi}}\Gamma^2\left(\frac{3}{4}\right)$$
$$f_2\left(\tfrac{1}{2}\right) = \int_0^1 \frac{\sqrt{x}\,\ln^2 x\,dx}{\sqrt{1-x^2}} = \frac{1}{2\sqrt{2\pi}}\Gamma^2\left(\frac{3}{4}\right)(32-16G+\pi(\pi-8)),$$

where G is **Catalan's constant**

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}. \tag{6.16}$$

Example 6.8. Entry **4.261.15** states that

$$\int_0^1 \frac{x^{2n}\,\ln^2 x}{\sqrt{1-x^2}}\,dx = \frac{(2n-1)!!}{2(2n)!!}\pi\left\{\frac{\pi^2}{12} + \sum_{k=1}^{2n}\frac{(-1)^k}{k^2} + \left[\sum_{k=1}^{2n}\frac{(-1)^k}{k} + \ln 2\right]^2\right\}. \tag{6.17}$$

This is obtained by differentiating $h(s)$ twice with respect to s to produce

$$\int_0^1 \frac{x^s\ln^2 x\,dx}{\sqrt{1-x^2}} = \frac{\sqrt{\pi}}{8}\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)}\left[\left(\psi\left(\frac{s}{2}+1\right)-\psi\left(\frac{s+1}{2}\right)\right)^2 + \psi'\left(\frac{s+1}{2}\right) - \psi'\left(\frac{s}{2}+1\right)\right].$$

Therefore

$$\int_0^1 \frac{x^{2n}\ln^2 x\,dx}{\sqrt{1-x^2}} = \frac{\sqrt{\pi}}{8}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\left[\left(\psi(n+1)-\psi\left(n+\tfrac{1}{2}\right)\right)^2 + \psi'\left(n+\tfrac{1}{2}\right) - \psi'(n+1)\right].$$

The special values

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} \text{ and } \Gamma(n+1) = n! \tag{6.18}$$

give

$$\int_0^1 \frac{x^{2n}\ln^2 x\,dx}{\sqrt{1-x^2}} = \frac{\pi}{8}\frac{(2n-1)!!}{(2n)!!}\left[\left(\psi(n+1)-\psi(n+\tfrac{1}{2}\right)^2 + \psi'(n+\tfrac{1}{2}) - \psi'(n+1)\right].$$

Now use the special values

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n}\frac{1}{k} \text{ and } \psi(n+\tfrac{1}{2}) = -\gamma - 2\ln 2 + 2\sum_{k=1}^{n}\frac{1}{2k-1} \tag{6.19}$$

as well as

$$\psi'(n+1) = \frac{\pi^2}{6} - \sum_{k=1}^{n}\frac{1}{k^2} \text{ and } \psi'(n+\tfrac{1}{2}) = \frac{\pi^2}{2} - 4\sum_{k=1}^{n}\frac{1}{(2k-1)^2} \tag{6.20}$$

to obtain

$$\psi(n+1) - \psi(n+\tfrac{1}{2}) = 2\sum_{k=1}^{2n} \frac{(-1)^k}{k} + 2\ln 2 \tag{6.21}$$

and

$$\psi'(n+\tfrac{1}{2}) - \psi'(n+1) = \frac{\pi^2}{3} + 4\sum_{k=1}^{2n} \frac{(-1)^k}{k^2}. \tag{6.22}$$

This gives the result.

Example 6.9. A similar analysis gives entry **4.261.16**

$$\int_0^1 \frac{x^{2n+1}\ln^2 x}{\sqrt{1-x^2}}\,dx =$$
$$-\frac{(2n)!!}{(2n+1)!!}\left\{\frac{\pi^2}{12} + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k^2} - \left[\sum_{k=1}^{2n+1}\frac{(-1)^k}{k} + \ln 2\right]^2\right\}.$$

Example 6.10. Entry **4.241.6** states that

$$\int_0^{1/\sqrt{2}} \frac{\ln x\,dx}{\sqrt{1-x^2}} = -\frac{\pi}{4}\ln 2 - \frac{G}{2}. \tag{6.23}$$

The change of variables $x = \sin t$ gives

$$\int_0^{1/\sqrt{2}} \frac{\ln x\,dx}{\sqrt{1-x^2}} = \int_0^{\pi/4} \ln\sin t\,dt. \tag{6.24}$$

This integral is entry **4.224.2** and it has been evaluated in [**3**].

7. An example producing a trigonometric answer

The next example contains, in the logarithmic part, a quotient of linear functions. The evaluation of this entry requires a different approach.

Example 7.1. Entry **4.297.8** states that

$$\int_0^1 \ln\frac{1+ax}{1-ax}\,\frac{dx}{x\sqrt{1-x^2}} = \pi\sin^{-1} a. \tag{7.1}$$

This evaluation starts with the expansion

$$\frac{1}{x}\,\ln\frac{1+ax}{1-ax} = \sum_{n=0}^{\infty}\frac{2a^{2n+1}}{2n+1}x^{2n} \tag{7.2}$$

to obtain

$$\int_0^1 \ln\frac{1+ax}{1-ax}\,\frac{dx}{x\sqrt{1-x^2}} = \sum_{n=0}^{\infty}\frac{2a^{2n+1}}{2n+1}\int_0^1\frac{x^{2n}\,dx}{\sqrt{1-x^2}}. \tag{7.3}$$

The change of variables $x = \sin\theta$ gives

$$\int_0^1 \frac{x^{2n}\,dx}{\sqrt{1-x^2}} = \int_0^{\pi/2} \sin^{2n}\theta\,d\theta = \frac{\pi}{2^{2n+1}}\binom{2n}{n}. \tag{7.4}$$

The last evaluation is the famous Wallis' formula. It appears as entry **3.621.3** and it was established in [**2**] and [**12**]. Therefore

$$\int_0^1 \ln\frac{1+ax}{1-ax}\,\frac{dx}{x\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{\pi}{2^{2n}}\,\frac{a^{2n+1}}{2n+1}\binom{2n}{n}. \tag{7.5}$$

The series is now identified from the classical expansion

$$\begin{aligned} \sin^{-1} x &= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{(2n+1)\,n!}x^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{1}{2^{2n}\,(2n+1)}\binom{2n}{n}x^{2n+1} \end{aligned}$$

obtained by expanding the integrand in

$$\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}. \tag{7.6}$$

as a binomial series and integrating term by term.

Further examples in [**5**], of the class considered here, will be presented in a future publication.

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