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The integrals in Gradshteyn and Ryzhik.
Part 29: Chebyshev polynomials

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Abstract. The table of Gradshteyn and Ryzhik contains many integrals that involve Chebyshev polynomials. Some examples are discussed.

1. Introduction

The Chebyshev polynomial of the first kind $T_n(x)$ is defined by the relation

$$\cos n\theta = T_n(\cos\theta). \tag{1.1}$$

The elementary recurrence

$$\cos(n+1)\theta = 2\cos\theta\cos n\theta - \cos(n-1)\theta \tag{1.2}$$

yields the three-term recurrence for orthogonal polynomials

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \tag{1.3}$$

and, with initial conditions $T_0(x)=1$ and $T_1(x)=x$, shows that $T_n(x)$ is indeed a polynomial in x. The polynomial $T_n(x)$ is of degree n and its leading coefficient is 2^{n-1}. These elementary facts follow directly from (1.3).

The Chebyshev polynomial of the second kind $U_n(x)$ is defined by the relation

$$\frac{\sin(n+1)\theta}{\sin\theta} = U_n(\cos\theta). \tag{1.4}$$

This polynomial satisfies the recurrence

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \tag{1.5}$$

(the same recurrence as (1.3)), this time with initial conditions $U_0(x)=1$ and $U_1(x)=2x$.

Some basic properties of Chebyshev polynomials are collected next. The first result gives the classical generating function for these polynomials.

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Proposition 1.1. The generating function for the Chebyshev polynomials is given by

$$\sum_{n=0}^{\infty} T_n(x)t^n = \frac{1-xt}{1-2xt+t^2} \tag{1.6}$$

and

$$\sum_{n=0}^{\infty} U_n(x)t^n = \frac{1}{1-2xt+t^2}. \tag{1.7}$$

Proof. Multiply the recurrence (1.3) by t^n and sum over $n \geqslant 1$. ☐

Binet's formula for Chebyshev polynomials follows directly from their generating functions (1.6) and (1.7).

Corollary 1.2. The Chebyshev polynomial $T_n(x)$ is given by

$$T_n(x) = \frac{1}{2}\left[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right]. \tag{1.8}$$

Similarly, the polynomial $U_n(x)$ is given by

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left[(x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}\right]. \tag{1.9}$$

Proof. Expand the right-hand side of (1.6) and (1.7) in partial fractions and expand the resulting terms. ☐

A useful expression for the Chebyshev polynomials is their Rodrigues formulas

$$T_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}\sqrt{1-x^2}\left(\frac{d}{dx}\right)^n (1-x^2)^{n-1/2} \tag{1.10}$$

and

$$U_n(x) = \frac{(-1)^n (n+1)! 2^n}{(2n+1)!}\frac{1}{\sqrt{1-x^2}}\left(\frac{d}{dx}\right)^n (1-x^2)^{n+1/2}. \tag{1.11}$$

These will be used in some simplifications in the rest of the paper.

2. Some elementary examples

The classical table of integrals [**2**] contains a small collection of integrals with $T_n(x)$ or $U_n(x)$ in the integrand. The goal of this note is to provide self-contained proofs of these entries. The most elementary entry is **7.343.1** that is equivalent to the orthogonality of the family $\{\cos n\theta\}$ on the interval $[0,\, 2\pi]$. Indeed, define

$$\langle f, g \rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\, dx, \tag{2.1}$$

then the first example simply gives $\langle T_n, T_m \rangle = 0$ if $n \neq m$.

Example 2.1. Entry 7.343.1:

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{\pi}{2} & \text{if } m = n \neq 0, \\ \pi & \text{if } m = n = 0. \end{cases} \tag{2.2}$$

The next couple of examples computes integrals involving powers of Chebyshev polynomials.

Example 2.2. The evaluation

$$\int_{-1}^{1} T_n(x)\,dx = \frac{(-1)^{n-1}-1}{(n-1)(n+1)}, \text{ for } n \geqslant 2, \tag{2.3}$$

is not included in [**2**]. To confirm this formula, let $x = \cos\theta$ and use the identity

$$\cos n\theta \, \sin\theta = \frac{1}{2}\left[\sin(n+1)\theta - \sin(n-1)\theta\right] \tag{2.4}$$

to produce

$$\int_{-1}^{1} T_n(x)\,dx = \frac{1}{2}\int_0^{\pi} \left[\sin(n+1)\theta - \sin(n-1)\theta\right]\,d\theta. \tag{2.5}$$

The result follows by computing the elementary trigonometric integrals.

The indefinite version of this entry appears in [**4**] as entry **1.14.2.1** in the form

$$\int T_n(x)\,dx = \frac{1}{2}\left[\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\right]. \tag{2.6}$$

To verify this evaluation, let $x = \cos\theta$ and observe that

$$\int T_n(x)\,dx = -\int \cos(n\theta)\sin\theta\,d\theta. \tag{2.7}$$

The result now follows from (2.4).

Example 2.3. Entry **7.341.1** states that

$$\int_{-1}^{1} T_n^2(x)\,dx = 1 - \frac{1}{4n^2-1} = \frac{2(2n^2-1)}{(2n-1)(2n+1)}. \tag{2.8}$$

The evaluation starts with (1.8) to obtain

$$\int_{-1}^{1} T_n^2(x)\,dx = \frac{1}{4}\int_{-1}^{1}(x+\sqrt{x^2-1})^{2n}\,dx + \frac{1}{4}\int_{-1}^{1}(x-\sqrt{x^2-1})^{2n}\,dx + 1. \tag{2.9}$$

The change of variables $x = \cos\theta$ gives

$$\int_{-1}^{1}(x+\sqrt{x^2-1})^{2n}\,dx = \int_0^{\pi} e^{2in\theta}\sin\theta\,d\theta. \tag{2.10}$$

The last integral is evaluated by writing $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$. The second integral is evaluated in the same manner and the stated formula is obtained from here. A generalization of this result is given in Section 8.

Example 2.4. Entry **7.341.2** is

$$\int_{-1}^{1} T_m(x)T_n(x)\,dx = \frac{1}{1-(m-n)^2} + \frac{1}{1-(m+n)^2} \quad \text{if } m+n \text{ is even} \tag{2.11}$$

and

$$\int_{-1}^{1} T_m(x)T_n(x)\,dx = 0 \quad \text{if } m+n \text{ is odd.} \tag{2.12}$$

The proof is based on the identity

$$T_n(x)T_m(x) = \frac{1}{2}\left[T_{n-m}(x) + T_{n+m}(x)\right] \tag{2.13}$$

coming from its trigonometric counterpart

$$\cos n\theta \cos m\theta = \frac{1}{2}\left[\cos(n+m)\theta + \cos(n-m)\theta\right]. \tag{2.14}$$

The result now follows from (2.6).

Example 2.5. The integral

$$\int (1-x^2)^{\frac{n-3}{2}} T_n(x)\,dx = -\frac{1}{n-1}(1-x^2)^{\frac{n-1}{2}} T_{n-1}(x) \tag{2.15}$$

appears as entry 1.14.2.3 in [**5**]. It does not appear in [**2**]. The proof is elementary: the change of variables $x = \cos\theta$ gives

$$\int (1-x^2)^{\frac{n-3}{2}} T_n(x)\,dx = -\int \sin^{n-2}\theta\,\cos n\theta\,d\theta \tag{2.16}$$

and the elementary identity

$$\sin^{n-2}\theta\,\cos n\theta = \frac{1}{n-1}\frac{d}{d\theta}\left[\sin^{n-1}\theta\,T_{n-1}(\cos\theta)\right]. \tag{2.17}$$

The companion entry 1.14.2.4 in [**5**]

$$\int (1-x^2)^{-\frac{n+3}{2}} T_n(x)\,dx = \frac{1}{n+1}(1-x^2)^{-\frac{n+1}{2}} T_{n+1}(x) \tag{2.18}$$

is established in a similar manner.

3. The evaluation of a Mellin transform

The Mellin transform of a function $f(x)$ is defined by

$$\mathcal{M}(f)(s) = \int_0^\infty x^{s-1} f(x)\,dx. \tag{3.1}$$

In examples concerning Chebyshev polynomials, with kernel $1/\sqrt{1-x^2}$, it is natural to consider their restriction to $[-1,1]$. Entry **7.346** states that

$$\int_0^1 x^{s-1} T_n(x) \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{s2^s}\left[B\left(\frac{1+s+n}{2}, \frac{1+s-n}{2}\right)\right]^{-1}, \text{ for } \operatorname{Re} s > 0. \tag{3.2}$$

This entry gives the Mellin transform of the function

$$f(x) = \begin{cases} T_n(x)/\sqrt{1-x^2} & \text{if } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

The change of variables $x = \cos\theta$ transforms (3.2) to

$$\int_0^{\pi/2} \cos(n\theta)\cos^{s-1}\theta\, d\theta = \frac{\pi}{s2^s}\left[B\left(\frac{1+s+n}{2}, \frac{1+s-n}{2}\right)\right]^{-1}. \tag{3.4}$$

This entry will be established in a future publication. Only a special case is required here.

Special Case. Assume $s = m+1$ is a positive integer. Then (3.4) becomes

$$\int_0^{\pi/2} \cos(n\theta)\cos^{m}\theta\, d\theta = \frac{\pi}{(m+1)2^{m+1}}\left[B\left(\frac{2+m+n}{2}, \frac{2+m-n}{2}\right)\right]^{-1}. \tag{3.5}$$

The reduction of this integral requires a simple trigonometric formula. This appears as entry **1.320** in [**2**]. The proof of (3.5) is presented next.

Lemma 3.1. *For $m \in \mathbb{N}$,*

$$x^m = \frac{1}{2^{m-1}} \sum_{k=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m}{k} T_{m-2k}(x) + \begin{cases} 0 & \text{if } m \equiv 1 \bmod 2 \\ 2^{-m}\binom{m}{m/2} & \text{if } m \equiv 0 \bmod 2. \end{cases} \tag{3.6}$$

Proof. Let $x = \cos\theta$ and start with

$$\begin{aligned} \cos^m\theta &= \frac{(e^{i\theta}+e^{-i\theta})^m}{2^m} \\ &= \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k} e^{i(2k-m)\theta}. \end{aligned} \tag{3.7}$$

By symmetry, since this a real function, the real part yields

$$\cos^m\theta = \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}\cos(m-2k)\theta. \tag{3.8}$$

To obtain the stated formula, split the sum in half to obtain

$$\cos^m\theta = \frac{1}{2^m}\sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor}\binom{m}{k}\cos(m-2k)\theta + \frac{1}{2^m}\sum_{k=\left\lfloor \frac{m}{2} \right\rfloor+1}^{m}\binom{m}{k}\cos(m-2k)\theta. \tag{3.9}$$

In the case m odd, both sums have the same number of elements and the change of indices $j = k - m/2$ shows that they are equal. In the case m even, there is an extra term corresponding to the index $m/2$. □

Then (3.2) gives

$$\int_0^1 x^m T_n(x)\frac{dx}{\sqrt{1-x^2}} = \frac{1}{2^{m-1}}\sum_{k=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor}\binom{m}{k}\int_0^1 \frac{T_n(x)T_{m-2k}(x)}{\sqrt{1-x^2}}\,dx \tag{3.10}$$

when m is odd and in the case m even there is the extra term producing

$$\int_0^1 x^m T_n(x) \frac{dx}{\sqrt{1-x^2}} = \frac{1}{2^{m-1}} \sum_{k=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m}{k} \int_0^1 \frac{T_n(x) T_{m-2k}(x)}{\sqrt{1-x^2}} \, dx + \frac{\binom{m}{m/2}}{2^m} \int_0^1 \frac{T_n(x) \, dx}{\sqrt{1-x^2}}. \tag{3.11}$$

Now consider the special case $m \equiv n \bmod 2$. The extra term coming when m is even now disappears because $n \geqslant 2$ is also even and

$$\int_0^1 \frac{T_n(x) \, dx}{\sqrt{1-x^2}} = \frac{1}{2} \int_{-1}^1 \frac{T_n(x) \, dx}{\sqrt{1-x^2}} = 0, \tag{3.12}$$

since $T_n(x)$ is orthogonal to $T_0(x) = 1$. For the remaining terms, observe that $T_n(x) T_{m-2k}(x)$ is an even polynomial and the integrals can be extended to $[-1, 1]$ to obtain

$$\int_0^1 x^m T_n(x) \frac{dx}{\sqrt{1-x^2}} = \frac{1}{2^m} \sum_{k=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m}{k} \int_{-1}^1 \frac{T_n(x) T_{m-2k}(x)}{\sqrt{1-x^2}} \, dx. \tag{3.13}$$

The orthogonality of Chebyshev polynomials implies that the integral in the summand vanishes unless $n = m - 2k$; that is, $k = \frac{1}{2}(m-n)$. If $m < n$ the integral on the left of (3.10) vanishes. This matches the right-hand side of (3.5), as the beta function value also vanishes. In the case $m \geqslant n$, it follows that

$$\int_0^1 x^m T_n(x) \frac{dx}{\sqrt{1-x^2}} = \frac{1}{2^m} \binom{m}{\frac{1}{2}(m-n)} \int_{-1}^1 \frac{T_n^2(x)}{\sqrt{1-x^2}} \, dx. \tag{3.14}$$

Now, for $n \geqslant 1$,

$$\int_{-1}^1 \frac{T_n^2(x)}{\sqrt{1-x^2}} \, dx = \int_0^\pi \cos^2(n\theta) \, d\theta = \frac{\pi}{2}, \tag{3.15}$$

and this produces

$$\int_0^1 \frac{x^m T_n(x)}{\sqrt{1-x^2}} \, dx = \frac{\pi}{2^{m+1}} \binom{m}{\frac{1}{2}(m-n)} = \frac{\pi}{2^{m+1}} \binom{m}{\frac{1}{2}(m+n)}. \tag{3.16}$$

This matches the answer given in (3.2).

Theorem 3.2. Let m, $n \in \mathbb{N}$. If $m \equiv n \bmod 2$

$$\int_0^1 \frac{x^m T_n(x)}{\sqrt{1-x^2}} \, dx = \frac{\pi}{2^{m+1}} \binom{m}{\frac{1}{2}(m+n)} \quad \text{if } m \geqslant n \tag{3.17}$$

and

$$\int_0^1 \frac{x^m T_n(x)}{\sqrt{1-x^2}} \, dx = 0 \quad \text{if } m < n. \tag{3.18}$$

Note 3.3. The reader is encouraged to verify that, when m and n have different parity, the integral is given by

$$\int_0^1 \frac{x^m T_n(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} \frac{2^{m-1} m! \left(\frac{m+n-1}{2}\right)! \left(\frac{m-n-1}{2}\right)!}{(m+n)!\,(m-n)!} & \text{if } m+1 > n, \\ (-1)^{(n-m-1)/2}\, \frac{2^m m! \left(\frac{m+n-1}{2}\right)! \,(n-m-1)!}{(m+n)! \left(\frac{n-m-1}{2}\right)!} & \text{if } m+1 \leqslant n. \end{cases} \tag{3.19}$$

4. A Fourier transform

This section describes entries in [**2**] that are related to the Fourier transform of the Chebyshev polynomials.

Entry **7.355.1**

$$\int_0^1 T_{2n+1}(x)\sin(ax)\frac{dx}{\sqrt{1-x^2}} = (-1)^n \frac{\pi}{2} J_{2n+1}(a) \tag{4.1}$$

and entry **7.355.2**

$$\int_0^1 T_{2n}(x)\cos(ax)\frac{dx}{\sqrt{1-x^2}} = (-1)^n \frac{\pi}{2} J_{2n}(a) \tag{4.2}$$

may be combined into the form

$$\int_{-1}^1 T_n(x) e^{iax} \frac{dx}{\sqrt{1-x^2}} = i^n \pi J_n(a), \tag{4.3}$$

where $J_\nu(z)$ is the Bessel function defined by

$$J_\nu(z) = \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{\nu+2k}. \tag{4.4}$$

This form appears as Entry **2.18.1.9** in [**4**]. Indeed, for $n = 2r$ even, the real part of (4.3) gives

$$\int_{-1}^1 T_{2r}(x)\cos(ax)\frac{dx}{\sqrt{1-x^2}} = (-1)^r \pi J_{2r}(a). \tag{4.5}$$

The expression (4.2) now comes from the parity of the integrand.

The proof of (4.3) begins with the change of variables $x = \cos\theta$ to produce

$$\int_{-1}^1 T_n(x) e^{iax} \frac{dx}{\sqrt{1-x^2}} = \int_0^\pi \cos(n\theta) e^{ia\cos\theta}\,d\theta. \tag{4.6}$$

Symmetry now gives

$$\begin{aligned} \int_0^\pi \cos(n\theta) e^{ia\cos\theta}\,d\theta &= \frac{1}{2}\int_0^\pi e^{i(-n\theta + a\cos\theta)}\,d\theta + \frac{1}{2}\int_0^\pi e^{i(n\theta + a\cos\theta)}\,d\theta \\ &= \frac{1}{2}\int_{-\pi}^0 e^{i(n\theta + a\cos\theta)}\,d\theta + \frac{1}{2}\int_0^\pi e^{i(n\theta + a\cos\theta)}\,d\theta \\ &= \frac{1}{2}\int_{-\pi}^\pi e^{i(n\theta + a\cos\theta)}\,d\theta. \end{aligned} \tag{4.7}$$

Aside from a scaling factor of 2π, this is the classical integral representation for the Bessel function

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ni\theta + iz\sin\theta} \, d\theta, \tag{4.8}$$

which is Entry **8.411.1** in [**2**].

An alternative proof of this entry uses Rodrigues formula for Chebyshev polynomials

$$T_n(x) = \frac{(-2)^n n!}{(2n)!} \sqrt{1-x^2} \frac{d^n}{dx^n} \left[(1-x^2)^{n-1/2} \right]. \tag{4.9}$$

Integrating by parts and using the fact that the boundary terms vanish yields

$$\begin{aligned} \int_{-1}^{1} \frac{T_n(x)}{\sqrt{1-x^2}} e^{ipx} \, dx &= \frac{(-2)^n n!}{(2n)!} \int_{-1}^{1} e^{ipx} \frac{d^n}{dx^n} \left[(1-x^2)^{n-1/2} \right] dx \\ &= \frac{2^n n!}{(2n)!} \int_{-1}^{1} (1-x^2)^{n-1/2} \frac{d^n}{dx^n} e^{ipx} \, dx \\ &= (ip)^n \frac{2^n n!}{(2n)!} \int_{-1}^{1} (1-x^2)^{n-1/2} e^{ipx} \, dx. \end{aligned}$$

Entry **3.771.8** implies that

$$\int_{-1}^{1} (1-x^2)^{n-1/2} e^{ipx} \, dx = \sqrt{\pi} \left(\frac{2}{p} \right)^n \Gamma\left(n + \tfrac{1}{2}\right) J_n(p), \tag{4.10}$$

which produces the result. A verification of (4.10), as well a many other entries in [**2**], will appear in a future publication.

A third proof of the present evaluation can be deduced from the operational formula given in the next lemma.

Lemma 4.1. The J-Bessel function of order n can be computed as

$$J_n(z) = i^n T_n \left(i \frac{d}{dz} \right) J_0(z) \tag{4.11}$$

where T_n is the Chebychev polynomial of the first kind.

Proof. Starting from the integral representation [**1**, 9.1.21]

$$J_n(z) = \frac{1}{\pi \imath^n} \int_0^{\pi} e^{\imath z \cos\theta} \cos(n\theta) \, d\theta,$$

we compute, with $T_n\left(z\right) = \sum_{k=0}^{n} t_{n,k} z^k$,

$$\begin{aligned}
\imath^n T_n\left(\imath \frac{d}{dz}\right) J_0\left(z\right) &= \frac{\imath^n}{\pi}\int_0^{\pi} T_n\left(\imath\frac{d}{dz}\right) e^{\imath z\cos\theta} d\theta \\
&= \frac{\imath^n}{\pi}\int_0^{\pi} \sum_{k=0}^{n} t_{n,k}\left(\imath\frac{d}{dz}\right)^k e^{\imath z\cos\theta} d\theta \\
&= \frac{\imath^n}{\pi}\int_0^{\pi} T_n\left(-\cos\theta\right) e^{\imath z\cos\theta} d\theta \\
&= \frac{\left(-\imath\right)^n}{\pi}\int_0^{\pi} \cos\left(n\theta\right) e^{\imath z\cos\theta} d\theta = J_n\left(z\right),
\end{aligned}$$

where the parity property $T_n(-x) = (-1)^n T_n(x)$ has been used. □

Using the former result and the Fourier identity

$$\int x^n f(x)\exp(-ipx)\,dx = \left(i\frac{d}{dp}\right)^n \hat{f}(p), \tag{4.12}$$

we deduce that, for any polynomial P,

$$\int P(x)f(x)\exp(-ipx)\,dx = P\left(i\frac{d}{dp}\right)\hat{f}(p). \tag{4.13}$$

Now use entry **3.753.2**

$$\int_{-1}^{1} \frac{\cos px\,dx}{\sqrt{1-x^2}} = \pi J_0(p) \tag{4.14}$$

to obtain

$$\int_{-1}^{1} \frac{T_n(x)}{\sqrt{1-x^2}}\cos(px)\,dx = T_n\left(i\frac{d}{dp}\right)\pi J_0(p) = \frac{\pi}{i^n} J_n(p). \tag{4.15}$$

5. An entry with two parameters

Section **7.342** consists of the single entry

$$\int_{-1}^{1} U_n\left[x(1-y^2)^{1/2}(1-z^2)^{1/2} + yz\right]\,dx = \frac{2}{n+1}U_n(y)U_n(z), \quad \text{for } |y|<1,\ |z|<1. \tag{5.1}$$

The parameters y, z can be expressed in trigonometric form by denoting

$$y = \cos\alpha, \quad z = \cos\beta \tag{5.2}$$

transforming (5.1) to

$$I := \int_{-1}^{1} U_n\left[x\sin\alpha\sin\beta + \cos\alpha\cos\beta\right]\,dx = \frac{2}{n+1}U_n(\cos\alpha)U_n(\cos\beta). \tag{5.3}$$

The basic relation among the two kinds of Chebyshev polynomials

$$\frac{d}{dx}T_n(x) = nU_{n-1}(x) \tag{5.4}$$

gives

$$\int U_n(ax+b)\,dx = \frac{1}{a(n+1)} T_{n+1}(ax+b). \tag{5.5}$$

Therefore

$$\begin{aligned}
(n+1)\sin\alpha\sin\beta \times I &= \left[T_{n+1}(x\sin\alpha\sin\beta + \cos\alpha\cos\beta)\right]\Big|_{x=-1}^{1} \\
&= T_{n+1}(\sin\alpha\sin\beta + \cos\alpha\cos\beta) - T_{n+1}(-\sin\alpha\sin\beta + \cos\alpha\cos\beta) \\
&= T_{n+1}(\cos(\alpha-\beta)) - T_{n+1}(\cos(\alpha+\beta)) \\
&= \cos[(n+1)(\alpha-\beta)] - \cos[(n+1)(\alpha+\beta)].
\end{aligned}$$

The elementary identity

$$\cos u - \cos v = -2\sin\frac{u+v}{2}\sin\frac{u-v}{2} \tag{5.6}$$

now produces

$$I = \frac{2}{n+1}\,\frac{\sin(n+1)\alpha}{\sin\alpha}\,\frac{\sin(n+1)\beta}{\sin\beta}. \tag{5.7}$$

This is the stated result.

6. An example involving Legendre polynomials

The integral

$$\int_a^b \frac{1}{\sqrt{(x-a)(b-x)}} T_n\left(\frac{x}{b}\right)\,dx = \frac{\pi}{2}\left[P_n\left(\frac{a}{b}\right) + P_{n-1}\left(\frac{a}{b}\right)\right], \tag{6.1}$$

where $b > a > 0$ and $P_n(x)$ is the Legendre polynomial, appears as entry **7.349** in [**2**] in the form

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(1-x^2y)\,dx = \frac{\pi}{2}\left[P_n(1-y) + P_{n-1}(1-y)\right]. \tag{6.2}$$

An automatic proof of this entry has been given in [**3**]. Its companion **7.348** is

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} U_{2n}(xz)\,dx = \pi P_n(2z^2-1), \quad |z| < 1. \tag{6.3}$$

The proof of (6.3) begins with the generating function

$$\sum_{n=0}^{\infty} U_n(x)t^n = \frac{1}{1-2xt+t^2}, \tag{6.4}$$

then dissection produces

$$\begin{aligned}
\sum_{n=0}^{\infty} U_{2n}(xz)t^{2n} &= \frac{1}{2}\left[\frac{1}{1-2xtz+t^2} + \frac{1}{1+2xtz+t^2}\right] \\
&= \frac{1}{(1+t^2)(1-a^2x^2)}
\end{aligned} \tag{6.5}$$

with

$$a = \frac{2tz}{1+t^2}. \tag{6.6}$$

Now observe that an elementary argument gives

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{dx}{1-a^2x^2} = \frac{1}{2}\int_0^{\pi} \frac{d\theta}{1+a\cos\theta} + \frac{1}{2}\int_0^{\pi} \frac{d\theta}{1-a\cos\theta} = \frac{\pi}{\sqrt{1-a^2}}, \tag{6.7}$$

since both integrals evaluate to $\pi/\sqrt{1-a^2}$. Replacing into (6.5) gives, after some elementary simplifications, the identity

$$\sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} U_{2n}(xz)\, dx = \frac{\pi}{\sqrt{(1+t^2)^2 - 4t^2z^2}}. \tag{6.8}$$

The result now follows from

$$\sum_{n=0}^{\infty} P_n(2z^2-1)t^{2n} = \frac{1}{\sqrt{(1+t^2)^2-4t^2z^2}}. \tag{6.9}$$

This last expression comes from the generating function

$$\sum_{k=0}^{\infty} t^k P_k(z) = \frac{1}{\sqrt{1-2tz+t^2}} \tag{6.10}$$

for the Legendre polynomials, given as entry **8.921** in [**2**].

7. A Hilbert transform

The two entries **7.344.1**

$$\int_{-1}^{1} \frac{1}{x-y}(1-y^2)^{-1/2} T_n(y)\, dy = -\pi U_{n-1}(x) \tag{7.1}$$

and **7.344.2**

$$\int_{-1}^{1} \frac{1}{x-y}(1-y^2)^{1/2} U_{n-1}(y)\, dy = \pi T_n(x) \tag{7.2}$$

are examples of the *Hilbert transform* defined by

$$\mathcal{H}(u)(x) = \frac{1}{\pi}\text{p.v.} \int_{-\infty}^{\infty} \frac{u(y)}{x-y}\, dy. \tag{7.3}$$

Actually, the integral in (7.1) has to be written as a principal value integral and x must be restricted to $-1 < x < 1$. Otherwise, the correct version of (7.1) is

$$\text{p.v.} \int_{-1}^{1} \frac{1}{x-y}(1-y^2)^{-1/2} T_n(y)\, dy = -\pi U_{n-1}(x) + \frac{h(x)}{\sqrt{x^2-1}} \pi T_n(x) \tag{7.4}$$

where

$$h(x) = \begin{cases} -1 & \text{if} \quad x < -1 \\ 0 & \text{if} \quad -1 < x < 1 \\ 1 & \text{if} \quad x > 1, \end{cases} \tag{7.5}$$

with a similar correction term for (7.2).

The evaluation of these entries uses the relation between the Fourier $\hat{u}$ and the Hilbert transform $\widehat{\mathcal{H}(u)}$ given by

$$\widehat{\mathcal{H}(u)}(\omega) = -\imath \operatorname{sign}(\omega)\, \widehat{u}(\omega). \tag{7.6}$$

Choosing

$$u(x) = \begin{cases} \frac{T_n(x)}{\sqrt{1-x^2}}, & \text{for } -1 < x < 1, \\ 0 & \text{otherwise,} \end{cases} \tag{7.7}$$

then (4.3) gives

$$\hat{u}(\omega) = \imath^n \pi J_n(\omega)$$

so that

$$\widehat{\mathcal{H}(u)}(\omega) = -\imath^{n+1}\pi \operatorname{sign}(\omega) J_n(\omega)$$

and the inverse Fourier transform is computed as

$$\mathcal{H}(u)(x) = \frac{1}{2\pi}\left[-\imath^{n+1}\pi \int_{-\infty}^{+\infty} \operatorname{sign}(\omega) J_n(\omega) e^{\imath\omega x} d\omega \right]. \tag{7.8}$$

The integral in (7.8) is written as

$$-\int_{-\infty}^{0} J_n(\omega) e^{\imath\omega x} d\omega + \int_0^\infty J_n(\omega) e^{\imath\omega x} d\omega = -\int_0^\infty \left(J_n(-\omega) e^{-\imath\omega x} - J_n(\omega) e^{\imath\omega x} \right) d\omega.$$

Each term is now computed using **6.611** in [**2**] to obtain

$$\int_0^\infty e^{-\alpha\omega} J_\nu(\beta\omega)\, d\omega = \frac{\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)^\nu}{\beta^\nu \sqrt{\alpha^2+\beta^2}}$$

to give

$$\int_0^\infty e^{\imath\omega x} J_n(\omega)\, d\omega = \imath^n \frac{\left(x+\sqrt{x^2-1}\right)^n}{\sqrt{1-x^2}}$$

and

$$\int_0^\infty e^{-\imath\omega x} J_n(-\omega)\, d\omega = \imath^n \frac{\left(x-\sqrt{x^2-1}\right)^n}{\sqrt{1-x^2}}$$

and it follows that

$$\begin{aligned} \mathcal{H}(u)(x) &= \frac{1}{2}\imath^{2n+1} \left[\frac{\left(x+\sqrt{x^2-1}\right)^n}{\sqrt{1-x^2}} - \frac{\left(x-\sqrt{x^2-1}\right)^n}{\sqrt{1-x^2}} \right] \\ &= \pi(-1)^{n-1} U_{n-1}(x). \end{aligned}$$

The result now follows from (7.3).

8. Integrals of powers

Entry **7.341** of [**2**] contains the entry

$$\int_{-1}^{1} T_n^2(x)\,dx = 1-(4n^2-1)^{-1} = \frac{4n^2-2}{4n^2-1}. \tag{8.1}$$

This has been described in Example 2.3 and it is a special case of the next result.

Theorem 8.1. *For $n,\, r \in \mathbb{N}$, the integral*

$$I_{n,r} = \int_{-1}^{1} T_n^r(x)\,dx \tag{8.2}$$

is given by

$$I_{n,r} = -\frac{(-1)^{nr}+1}{2^r}\sum_{\ell=0}^{r}\frac{\binom{r}{\ell}}{n^2(2\ell-r)^2-1}. \tag{8.3}$$

In particular, aside from an elementary factor, the integral $I_{n,r}$ is a rational function in the variable $x=n^2$.

Proof. Using the representation

$$T_n(x) = \frac{1}{2}\left[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right] \tag{8.4}$$

the integral becomes, after the change $x=\cos\theta$,

$$I_{n,r} = \frac{1}{2^r}\sum_{\ell=0}^{r}\binom{r}{\ell}\int_0^{\pi} e^{in\theta\ell}e^{-in\theta(r-\ell)}\sin\theta\,d\theta. \tag{8.5}$$

Now use the expression of $\sin\theta$ in terms of complex exponentials to obtain

$$I_{n,r} = \frac{1}{i2^{r+1}}\sum_{\ell=0}^{r}\binom{r}{\ell}\int_0^{\pi}\left(e^{i\theta(n(2\ell-r)+1)} - e^{i\theta(n(2\ell-r)-1)}\right)d\theta. \tag{8.6}$$

The result now follows by direct integration. ☐

Remark 8.1. The rational function mentioned above has intriguing arithmetic properties. These will be described in a future publication.

The expression for $I_{n,r}$ given above is now written in hypergeometric form. An elementary proof comes from writing the hypergeometric sum and using

$$(-r)_m = \frac{(-1)^m\,r!}{(r-m)!}. \tag{8.7}$$

Lemma 8.1. *For $n,\, r\in\mathbb{N}$, one has*

$$\sum_{\ell=0}^{r}\binom{r}{\ell}\frac{1}{n(2\ell-r)+1} = \frac{1}{1-nr}\,{}_2F_1\left(\left.\begin{matrix}\frac{1-nr}{2n},\, -r\\ 1+\frac{1-nr}{2n}\end{matrix}\right|-1\right) \tag{8.8}$$

and

$$\sum_{\ell=0}^{r}\binom{r}{\ell}\frac{1}{n(2\ell-r)-1} = \frac{1}{1+nr}\,{}_2F_1\left(\left.\begin{matrix}-\frac{1+nr}{2n},\, -r\\ 1-\frac{1+nr}{2n}\end{matrix}\right|-1\right). \tag{8.9}$$

The hypergeometric sum appearing in the previous lemma is given in [**5**, volume 3, 7.3.5.18] in terms of the Jacobi polynomials:

$$ {}_2F_1\left(\begin{matrix}-r, b\\ c\end{matrix}\middle| -1\right) = \frac{r!(-2)^r}{(c)_r} P_r^{(-b-r, c-1)}(0). \tag{8.10}$$

Therefore, the integral $I_{n,r}$ is now expressed in terms of Jacobi polynomials.

Theorem 8.1. Let $n,\ r \in \mathbb{N}$. The integral of a power of the Chebyshev polynomial of the first kind

$$ I_{n,r} = \int_{-1}^{1} T_n^r(x)\,dx \tag{8.11}$$

is given in terms of the Jacobi polynomial

$$ P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n}\sum_{j=0}^{n}\binom{n+\alpha}{j}\binom{n+\beta}{n-j}(x-1)^{n-j}(x+1)^j \tag{8.12}$$

by

$$ I_{n,r} \quad = \quad (-1)^r r!\, \frac{1+(-1)^{nr}}{4n}\left[\frac{(1+\alpha)_r^{-1}}{\alpha}P_r^{(\beta,\alpha)}(0) - \frac{(1+\beta)_r^{-1}}{\beta}P_r^{(\alpha,\beta)}(0)\right],$$

with $\alpha = \frac{1-rn}{2n}$ and $\beta = -\frac{1+rn}{2n}$.

Note 8.2. It is an interesting question to develop similar formulas for the integral

$$ J_{n,r} = \int_{-1}^{1} U_n^r(x)\,dx. \tag{8.13}$$

This is left to the interested reader.

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