THE DEFINITION OF BERNOULLI NUMBERS

The exponential function

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{1}$$

is one of the basic functions. Observe that $e^x - 1$ vanishes at $x = 0$, so the function

$$f(x) = \frac{e^x - 1}{x} \tag{2}$$

has a nice expansion at $x = 0$ given by

$$f(x) = \frac{e^x - 1}{x} = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}. \tag{3}$$

The question considered here is the coefficients in the expansion of the `reciprocal function`

$$g(x) = \frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}, \tag{4}$$

called the **Bernoulli numbers**.

Multiplying the series for f and its reciprocal will give a recurrence for the Bernoulli numbers:

$$\begin{aligned} 1 &= \left(\sum_{k=0}^{\infty} \frac{x^k}{(k+1)!} \right) \times \left(\sum_{j=0}^{\infty} B_j \frac{x^j}{j!} \right) \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{B_j}{j!(k+1)!} x^{j+k} \\ &= \sum_{r=0}^{\infty} \left(\sum_{j=0}^{r} \frac{B_j}{j!(r+1-j)!} \right) x^r, \end{aligned}$$

and we conclude that $B_0 = 1$ and

$$\sum_{j=0}^{r} \frac{B_j}{j!\,(r+1-j)!} = 0, \qquad \text{for } r \geq 1. \tag{5}$$

This last relation may be written as

$$\sum_{j=0}^{r} \binom{r+1}{j} B_j = 0. \tag{6}$$

Solving for the term B_r gives the recurrence

$$B_r = -\frac{1}{r+1} \sum_{j=0}^{r-1} \binom{r+1}{j} B_j. \tag{7}$$

It follows directly from here that B_r `is a rational number`.

The recurrence (7) gives $B_1 = -\frac{1}{2}$. It turns out that this is `the only non-vanishing odd Bernoulli number`. This follows directly from the expansion

$$\begin{aligned} g(x) - 1 + \frac{x}{2} &= \frac{x}{e^x - 1} - 1 + \frac{x}{2} \\ &= x\left(\frac{e^x + 1}{e^x - 1}\right) - 1 \\ &= x\left(\frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}}\right) - 1. \end{aligned}$$

The Bernoulli numbers appear in several series expansion. For instance, observe that

$$\begin{aligned} \cot u &= \frac{\cos u}{\sin u} \\ &= i\frac{e^{iu} + e^{-iu}}{e^{iu} - e^{-iu}} = i\frac{e^{2iu} + 1}{e^{2iu} - 1} = i\left(1 + \frac{2}{e^{2iu} - 1}\right). \end{aligned} \tag{8}$$

The relation (8) implies

$$\cot u = i\left(1 + \frac{g(2iu)}{iu}\right) = i\left(1 + \frac{1}{iu}\sum_{n=0}^{\infty}\frac{B_n}{n!}(2iu)^n\right).$$

Since $B_1 = -1/2$ is the only odd Bernoulli number, we obtain

$$\cot u = i\left(1 + \frac{1}{iu}\left[\sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}(2iu)^{2n} - \left(\frac{1}{2}\right)(2iu)^1\right]\right). \tag{9}$$

This gives

$$\cot u = \sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}(-1)^n 2^{2n} u^{2n-1}. \tag{10}$$

Using the trigonometric identity

$$\tan u = \cot u - 2\cot(2u) \tag{11}$$

gives the expansion

$$\tan u = \sum_{n=0}^{\infty}\frac{(-1)^{n-1}B_{2n}}{(2n)!}2^{2n}\left(2^{2n} - 1\right)u^{2n-1}. \tag{12}$$

The sign of the Bernoulli numbers is easy to discover. A nice of proof of the correct expression for them was given by Mordell [1]. It is based on the identity

$$\frac{x}{e^x + 1} = \frac{x}{e^x - 1} - \frac{2x}{e^{2x} - 1} = -\sum_{a=1}^{\infty}\left(2^a - 1\right)B_a x^a. \tag{13}$$

Multiply this relation by $x/(e^x - 1)$ to obtain the recurrence

$$B_{2a} = -\sum_{r=0}^{a-1}\frac{2^{2r} - 1}{2^{2a} - 1}\binom{2a}{2r}B_{2r}B_{2a-2r}. \tag{14}$$

Introduce the notation

$$b_a = (-1)^{a-1}B_{2a} \tag{15}$$

and then (14) becomes

$$b_a = \sum_{r=1}^{a-1} \frac{2^{2r}-1}{2^{2a}-1} \binom{2a}{2r} b_r b_{a-r}. \tag{16}$$

The initial value is $b_1 = B_2 > 0$ and the positivity of b_a is propagated from this last recurrence.

Now we know that b_a is a positive rational number. On a separate note you will find a proof of a result of von Staudt and Clausen that determines a formula for its denominator: the denominator of B_{2n} is the product of all the primes p such that $p-1$ divides $2n$. For example the primes $p=2$ and $p=3$ always appear as factors of the denominator. The proof is not so simple. The structure of the numerator of B_{2n} is even more mysterious.

References

[1] L. J. Mordell. The sign of the Bernoulli numbers. *Amer. Math. Monthly*, 80:547–548, 1973.