THE PARITY OF CATALAN NUMBERS

The question is :

When is the Catalan number an odd number?

Computing the first (say 20) values leads to the conjecture that C_n is odd precisely when n has the form $2^a - 1$.

To prove this, recall the recurrence for the Catalan numbers

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

The discussion now uses the symmetry of this sum in the index k. Observe that the sum has $n+1$ terms.

`Case 1`. Assume n is odd. Then there are an even number of terms and the sum is written as

$$C_{n+1} = 2 \sum_{k=0}^{\frac{n-1}{2}} C_k C_{n-k}.$$

This shows C_{n+1} is an even number. Therefore, if C_{n+1} is an odd number, then n must be even.

`Case 2`. Assume n is even. Then the sum has an odd number of terms. It can be written as

$$\begin{aligned} C_{n+1} &= \sum_{k=0}^{n} C_k C_{n-k} \\ &= \sum_{k=0}^{\frac{n}{2}-1} C_k C_{n-k} + C_{n/2}^2 + \sum_{k=\frac{n}{2}+1}^{n} C_k C_{n-k}. \end{aligned}$$

The change of indices $j = n - k$ shows that the last two sums and we have

$$C_{n+1} = 2 \sum_{k=0}^{\frac{n}{2}-1} C_k C_{n-k} + C_{n/2}^2.$$

and since squaring does not change parity, this proves that for n even: C_{n+1} is odd if and only if $C_{n/2}$ is odd. Write $m = n+1$, to convert the previous statement to:

`Assume` m `is odd. Then` C_m `is odd if and only if` $C_{\frac{m-1}{2}}$ `is odd.`

Date: October 2, 2019.

Write m odd in binary as

$$m = 1\, a_1\, a_2\, \cdots a_r\, 1.$$

Then

$$\frac{m-1}{2} = 1\, a_1\, a_2 \cdots a_r.$$

Therefore, if C_m is odd, it follows that $a_r = 1$. Continuing this process shows that all the binary digits of m must be 1. This is equivalent to the fact that m has the form $2^a - 1$, as claimed.