FIBONACCI NUMBERS

The original question is this:

How many sequences of 1 *and* 2 *add up to* n*?*

The problem can be redefined in terms of tilings: we have a rectangle of size $n \times 1$ that we want to cover using squares (tiles of size 1×1) and dominos (tiles of size 2×1).
Check that both problems are really the same.

We introduce the *counting function*
$a_n =$ the number of ways that this can be done.

Looking at the first tile, we divide the count in two disjoint classes: those that start with a square and those that start with a domino. The *addition principle* shows that

$$a_n = a_{n-1} + a_{n-2}, \quad \text{for } n \geq 3.$$

Now we introduce the **generating function** of the sequence $\{a_n\}$ by the expression

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Now multiply the recurrence satisfied by the a_n by x^n and sum for $n = 2, 3, 4, \cdots$ to get

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n.$$

Now we simplify these expressions to relate them to the generating function F. First

$$\sum_{n=2}^{\infty} a_n x^n = F(x) - a_0 - a_1 x$$

(The two first terms are missing on the left),

then

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$$\begin{aligned}\sum_{n=2}^{\infty} a_{n-1}x^n &= x\sum_{n=2}^{\infty} a_{n-1}x^{n-1}\\ &= x\sum_{n=1}^{\infty} a_n x^n\\ &= x\left(F(x)-a_0\right).\end{aligned}$$

Think about these two steps, they are very important.

The result of this is an expression for F:

$$F(x) = \frac{1}{1-x-x^2}.$$

Now recall that a_n is the coefficient of the expansion of F that multiplies x^n. In calculus one has the formula

$$g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}x^n$$

so we can get the a_n by differentiating F a number of times ($= n$) and then putting $x = 0$. *This is hard.* Check it.

The better way is to express F in *partial fractions*. This is what you do:

solve $1 - x - x^2 = 0$ to get two solutions

$$x_+ = -\frac{1}{2}(1+\sqrt{5}) \quad \text{and} \quad x_- = \frac{1}{2}(-1+\sqrt{5})$$

so that

$$1 - x - x^2 = -(x-x_+)(x-x_-).$$

Then we write

$$\frac{1}{1-x-x^2} = \frac{A}{x-x_+} + \frac{B}{x-x_-}$$

for some constants A and B. You have to find these constants (simply add the fractions back) and you get

$$F(x) = \frac{1}{\sqrt{5}}\left(\frac{1}{x-x_+} - \frac{1}{x-x_-}\right).$$

To expand these functions in power series is easy:

$$\frac{1}{x-x_+} = -\frac{1}{x_+}\frac{1}{1-x/x_+}$$

and now recall the only thing about series that you need in this course: the **geometric series**:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.$$

To simplify the answer even further, note that $x_+ \times x_- = -1$, so the formula is

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

In particular, when n is very large

$$a_n \sim \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}.$$

The numbers a_n are called **Fibonacci numbers**. They will come back.