# THE EXPANSION OF BERNOULLI POLYNOMIALS IN FOURIER SERIES

This note contains the details of the expansion in Fourier series of $B_n(x)$.

Assume that the function $f(x)$ is periodic of period $T$. Then it is determined by its values on the interval $[-T/2,\, T/2]$. Under some simple hypothesis, the function admits an expansion of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{T}\right). \tag{1}$$

The coefficients $a_n$ and $b_n$ are called the *Fourier coefficients* of $f$.

To evaluate the coefficients we use the *orthogonality* of the functions sin and cos that appear in (1). This simply means that

$$\int_{-T/2}^{T/2} \cos\left(\frac{2\pi n x}{T}\right) \sin\left(\frac{2\pi m x}{T}\right) dx = 0 \tag{2}$$

for all $n,\, m \in \mathbb{N}$, also

$$\int_{-T/2}^{T/2} \cos\left(\frac{2\pi n x}{T}\right) \cos\left(\frac{2\pi m x}{T}\right) dx = 0 \tag{3}$$

and

$$\int_{-T/2}^{T/2} \sin\left(\frac{2\pi n x}{T}\right) \sin\left(\frac{2\pi m x}{T}\right) dx = 0 \tag{4}$$

for $n,\, m \in \mathbb{N}$ and $m \neq n$ and finally

$$\int_{-T/2}^{T/2} \sin^2\left(\frac{2\pi n x}{T}\right) dx = \int_{-T/2}^{T/2} \cos^2\left(\frac{2\pi n x}{T}\right) dx = \frac{T}{2}. \tag{5}$$

To evaluate the coefficient $b_r$, multiply (1) by $\sin(2\pi r x/T)$ and integrate over the interval $[-T/2,\, T/2]$. All the resulting integral vanish except the one corresponding to the index $r$. This gives

$$b_r = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \sin\left(\frac{2\pi r x}{T}\right) dx \quad \text{for } r \geq 1. \tag{6}$$

Similarly

$$a_r = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2\pi r x}{T}\right) dx \quad \text{for } r \geq 1. \tag{7}$$

The coefficient $a_0$ is special: its formula is

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(x)\, dx. \tag{8}$$

The goal is to compute the Fourier expansion of the Bernoulli polynomials. We start with

$$B_1(x) = x - \tfrac{1}{2}. \tag{9}$$

Of course these are not periodic functions, so what I mean is to take the function $B_1(x)$ over a certain interval and then extend in a periodic form.

To start, take

$$f(x) = x \quad \text{on } \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]. \tag{10}$$

In this case $T = 1$ and the Fourier coefficients are

$$a_n = 0, \text{ for all } n \geq 0 \tag{11}$$

because $f(x)$ is odd and

$$b_n = 2\int_{-1/2}^{1/2} x \sin(2\pi n x)\, dx = 4\int_0^{1/2} x\sin(2\pi n x)\, dx = \frac{(-1)^{n+1}}{\pi n} \tag{12}$$

for $n \geq 1$. Therefore the Fourier expansion is

$$x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} \sin(2\pi n x), \qquad \text{for } -\tfrac{1}{2} < x < \tfrac{1}{2}. \tag{13}$$

Now shift to the interval $[0, 1]$ using $y = x - 1/2$ (and then writing $x$ instead of $y$) to obtain

$$x - \tfrac{1}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} \sin\left[2\pi n(x - \tfrac{1}{2})\right], \qquad \text{for } 0 < x < 1. \tag{14}$$

Note that one has to be careful with the continuity issues at the end of the interval: at $x = 0$ the left-hand side of (14) becomes $-1/2$ and the right-hand side gives 0.

The left-hand side of (14) is the first Bernoulli polynomial. To make it periodic, recall the *fractional part* of $x$, defined by

$$\{x\} = x - [x], \tag{15}$$

where $[x]$ is the *integer part* of $x$; this is the largest integer less or equal than $x$. Then (14) becomes

$$B_1(\{x\}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} \sin\left[2\pi n(x - \tfrac{1}{2})\right], \qquad \text{for } x \in \mathbb{R}. \tag{16}$$

Now go back to the interval $[0, 1]$ and write (16) in the form

$$B_1(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} \sin\left[2\pi n(x - \tfrac{1}{2})\right], \qquad \text{for } x \in [0, 1]. \tag{17}$$

Recall the basic property

$$B_n'(x) = nB_{n-1}(x) \tag{18}$$

that gives

$$B_2'(x) = 2B_1(x) \tag{19}$$

and (17) now becomes

$$\frac{1}{2}B_2'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} \sin\left[2\pi n(x - \tfrac{1}{2})\right], \qquad \text{for } x \in [0, 1]. \tag{20}$$

Integrate to obtain

$$\frac{1}{2}B_2(x) = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} \frac{\cos\left[2\pi n(x-\frac{1}{2})\right]}{2\pi n} + C_2, \tag{21}$$

where $C_2$ is a constant of integration. The constant of integration is obtained from the normalization

$$\int_0^1 B_n(x)\,dx = 0 \quad \text{for all } n \geq 1. \tag{22}$$

Using

$$\int_0^1 \cos\left[2\pi n\left(x-\tfrac{1}{2}\right)\right]\,dx = \int_{-1/2}^{1/2} \cos\left[2\pi n t\right]\,dt = \frac{1}{2\pi n}\int_{-\pi n}^{\pi n} \cos s\,ds = 0 \tag{23}$$

gives $C_2 = 0$ and (21) becomes

$$B_2(x) = -2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} \frac{\cos\left[2\pi n(x-\frac{1}{2})\right]}{2\pi n}. \tag{24}$$

This can be written in the form

$$B_2(x) = -4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^2} \cos\left[2\pi n(x-\tfrac{1}{2})\right], \quad \text{for } 0 < x < 1. \tag{25}$$

The series in (27) converges uniformly because

$$\left| -4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^2} \cos\left[2\pi n(x-\tfrac{1}{2})\right] \right| \leq \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \tag{26}$$

and the uniform convergence follows from Weierstrass M-test. Therefore it is valid to evaluate both sides at an interior point.

To get an idea of what is coming, observe that $x = \frac{1}{2}$ in (25) gives

$$B_2\left(\frac{1}{2}\right) = -4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^2}. \tag{27}$$

Now recall that

$$B_2(x) = x^2 - x + \frac{1}{6} \tag{28}$$

and therefore

$$B_2\left(\frac{1}{2}\right) = -\frac{1}{12}. \tag{29}$$

In this form, equation (27) becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}. \tag{30}$$

In order to reduce (30) to a more familiar form, split the index $n$ in the series according to parity to obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k)^2} \tag{31}$$

and use

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \left(1 - \frac{1}{4}\right) \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{3}{4} \sum_{k=1}^{\infty} \frac{1}{k^2}. \tag{32}$$

Replace in (31) to obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{3}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} + \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} = -\frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2}. \tag{33}$$

Replace this in (30) to produce

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{34}$$

Now go back to (25) (that I am copying here to make it easier to read)

$$B_2(x) = -4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^2} \cos\left[2\pi n(x - \tfrac{1}{2})\right], \quad \text{for } 0 < x < 1 \tag{35}$$

and use the relation (18) with $n = 3$ to get

$$B_3'(x) = 3B_2(x). \tag{36}$$

Integrate to obtain

$$B_3(x) = -12 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^2} \frac{\sin\left[2\pi n(x - \frac{1}{2})\right]}{2\pi n} + C_3 \tag{37}$$

for some constant of integration $C_3$. To obtain the value of $C_3$ use the analogue of (23) in the form

$$\int_0^1 \sin\left[2\pi n\left(x - \tfrac{1}{2}\right)\right] dx = \int_{-1/2}^{1/2} \sin\left[2\pi n t\right] dt = \frac{1}{2\pi n} \int_{-\pi n}^{\pi n} \sin s \, ds = 0 \tag{38}$$

and conclude that $C_3 = 0$. Therefore

$$B_3(x) = -2 \cdot 3! \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^3} \sin\left[2\pi n(x - \tfrac{1}{2})\right]. \tag{39}$$

This time, replacing $x = \frac{1}{2}$ simply gives

$$B_3\left(\tfrac{1}{2}\right) = 0. \tag{40}$$

This is clear from

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{x}{2}. \tag{41}$$

Now compare the forms

$$B_2(x) = -2 \cdot 2! \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^2} \cos\left[2\pi n(x - \tfrac{1}{2})\right]. \tag{42}$$

and

$$B_3(x) = -2 \cdot 3! \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^3} \sin\left[2\pi n(x - \tfrac{1}{2})\right] \tag{43}$$

to guess a general pattern.

Now use the relation (18) with $n = 4$ to get

$$B_4'(x) = 4B_3(x) \tag{44}$$

and integrate (39) to get

$$B_4(x) = 2 \cdot 4! \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^4} \cos\left[2\pi n(x - \tfrac{1}{2})\right] \tag{45}$$

where the constant of integration vanishes as before.

Iterating this process leads to

$$B_{2k}(x) = 2(-1)^k \cdot (2k)! \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^{2k}} \cos\left[2\pi n(x - \tfrac{1}{2})\right] \tag{46}$$

To prove this result by induction, use

$$B_{2k+1}'(x) = (2k+1)B_{2k}(x) \tag{47}$$

and integrate (46) to produce

$$\begin{aligned} B_{2k+1}(x) &= (2k+1) \times 2(-1)^k \cdot (2k)! \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^{2k}} \frac{1}{2\pi n} \sin\left[2\pi n(x - \tfrac{1}{2})\right] \\ &= 2(-1)^k \cdot (2k+1)! \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^{2k+1}} \sin\left[2\pi n(x - \tfrac{1}{2})\right], \end{aligned}$$

and integrating

$$B_{2k+2}'(x) = (2k+2)B_{2k+1}(x) \tag{48}$$

to get

$$B_{2k+2}(x) = 2(-1)^{k+1} \cdot (2k+2)! \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^{2k+2}} \cos\left[2\pi n(x - \tfrac{1}{2})\right] \tag{49}$$

This proves (46) by induction.

Using (46) yields

$$\begin{aligned} B_{2k+1}'(x) &= (2k+1)B_{2k}(x) \\ &= 2(-1)^k \cdot (2k+1)! \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^{2k+1}} \cos\left[2\pi n(x - \tfrac{1}{2})\right] \end{aligned}$$

Now integrate to get

$$B_{2k+1}(x) = 2(-1)^k \cdot (2k+1)! \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^{2k+1}} \sin\left[2\pi n(x - \tfrac{1}{2})\right] \tag{50}$$

This is summarized in the next statement. As before the extension of the polynomial $P(x)$ is given by $P(\{x\})$.

**Theorem 1.** *The Fourier series for the periodic extensions of the Bernoulli polynomials is given by*

$$B_{2k}(\{x\}) = 2(-1)^k \cdot (2k)! \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^{2k}} \cos\left[2\pi n(x - \tfrac{1}{2})\right] \tag{51}$$

*and*

$$B_{2k+1}(\{x\}) = 2(-1)^k \cdot (2k+1)! \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^{2k+1}} \sin\left[2\pi n(x - \tfrac{1}{2})\right]. \tag{52}$$

Now replace $x = 0$ in (51) to obtain

$$B_{2k}(0) = 2(-1)^{k-1} \cdot (2k)! \sum_{n=1}^{\infty} \frac{1}{(2\pi n)^{2k}} \tag{53}$$

This is now written in a more familiar form. Recall the form of the Bernoulli polynomial

$$B_r(x) = \sum_{j=0}^{r} \binom{r}{j} B_j x^{r-j} \tag{54}$$

and, using $x = 0$, gives

$$B_r(0) = B_r. \tag{55}$$

Therefore (53) is written as

$$B_{2k} = \frac{2(-1)^{k-1} \cdot (2k)!}{(2\pi)^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}}. \tag{56}$$

**Definition 2.** The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{57}$$

**Theorem 3.** *The special value of the Riemann zeta function $\zeta(2k)$ at an even integer is a rational multiple of $\pi^{2k}$. The explicit expression is given by*

$$\zeta(2k) = \frac{2^{2k-1}}{(2k)!}(-1)^{k-1} B_{2k} \times \pi^{2k} \tag{58}$$

**Corollary 4.** *The sign of the Bernoulli number $B_{2k}$ is $(-1)^{k-1}$.*