A FORMULA FOR FRANEL NUMBERS

The `Franel numbers` are defined by

$$\texttt{Fra}_n = \sum_{k=0}^{n} \binom{n}{k}^3. \tag{1}$$

The goal of this section is to present a formula for these numbers in terms of the Jacobi polynomial. The notes presented here follow the presentation in [2].

The Jacobi polynomial is defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{(n+\alpha+\beta+1)_k}{(\alpha+1)_k} \left(\frac{x-1}{2} \right)^k. \tag{2}$$

Here $(t)_n$ is the Pochhammer symbol, defined by

$$(t)_j = t(t+1)(t+2)\cdots(t+j-1). \tag{3}$$

The first step is to produce an expression of the Jacobi polynomial in terms of the hypergeometric function.

Lemma 1. The Jacobi polynomial is given by

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} {}_2F_1 \left(\begin{matrix} -n & n+\alpha+\beta+1 \\ & \alpha+1 \end{matrix} \middle| \frac{1-x}{2} \right). \tag{4}$$

Proof. The hypergeometric function is given by

$${}_2F_1 \left(\begin{matrix} -n & n+\alpha+\beta+1 \\ & \alpha+1 \end{matrix} \middle| \frac{1-x}{2} \right) = \sum_{k=0}^{\infty} \frac{(-n)_k (n+\alpha+\beta+1)_k}{(\alpha+1)_k \, k!} \frac{(1-x)^k}{2^k}$$

and since $(-n)_k = 0$ for $k \geq n+1$, the sum is finite

$${}_2F_1 \left(\begin{matrix} -n & n+\alpha+\beta+1 \\ & \alpha+1 \end{matrix} \middle| \frac{1-x}{2} \right) = \sum_{k=0}^{n} (-1)^k \frac{(-n)_k (n+\alpha+\beta+1)_k}{(\alpha+1)_k \, k!} \left(\frac{x-1}{2} \right)^k,$$

and the identity is equivalent to

$$\frac{(\alpha+1)_n}{n!} \binom{n}{k} \frac{(n+\alpha+\beta+1)_k}{(\alpha+1)_k} = \binom{n+\alpha}{n} \frac{(-1)^k (-n)_k (n+\alpha+\beta+1)_k}{(\alpha+1)_k \, k!}$$

that reduces to

$$\frac{(\alpha+1)_n}{(n-k)!} = (-1)^k (-n)_k \binom{n+\alpha}{n}. \tag{5}$$

Date: September 26, 2019.

This is established by using the rules

$$(-n)_k = (-1)^k \frac{n!}{(n-k)!} \tag{6}$$

and

$$(x)_j = \frac{\Gamma(x+j)}{\Gamma(x)} \quad \text{and} \quad \binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\,\Gamma(u-v+1)}. \tag{7}$$

□

The next step is to produce a second form of the Jacobi polynomials. The proof is based on a transformation for the hypergeometric function. This appears as entry 9.131.1 in the table of integrals [1].

Lemma 2. The hypergeometric function satisfies

$$ {}_2F_1\left(\begin{matrix}\alpha & \beta \\ & \gamma\end{matrix}\middle| z\right) = (1-z)^{-\alpha} {}_2F_1\left(\begin{matrix}\alpha & \gamma-\beta \\ & \gamma\end{matrix}\middle| \frac{z}{z-1}\right) \tag{8}$$

Proof. Start with the integral definition of the hypergeometric function

$$ {}_2F_1\left(\begin{matrix}\alpha & \beta \\ & \gamma\end{matrix}\middle| z\right) = \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt \tag{9}$$

valid for $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$ (appearing as entry 9.111. in [1]). Now write

$$\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\left(1 - t\frac{z}{z-1}\right)^{-\alpha} dt = \\ (1-z)^{\alpha}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-z+tz)^{-\alpha}\,dt$$

and then make the change of variables $s = 1-t$ to obtain the result. □

Now use this identity with

$$z = \frac{1-x}{2},\ \alpha \mapsto -n,\ \beta \mapsto n+\alpha+\beta+1,\ \gamma \mapsto \alpha+1 \tag{10}$$

to obtain

$$ {}_2F_1\left(\begin{matrix}-n & n+\alpha+\beta+1 \\ & \alpha+1\end{matrix}\middle| \frac{1-x}{2}\right) = \left(\frac{1+x}{2}\right)^n {}_2F_1\left(\begin{matrix}-n & -n-\beta \\ & \alpha+1\end{matrix}\middle| \frac{x-1}{x+1}\right).$$

Lemma 1 now gives

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}\left(\frac{1+x}{2}\right)^n {}_2F_1\left(\begin{matrix}-n & -n-\beta \\ & \alpha+1\end{matrix}\middle| \frac{x-1}{x+1}\right). \tag{11}$$

Writing this in detail produces

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \binom{n+\alpha}{n} \left(\frac{1+x}{2}\right)^n {}_2F_1\left(\begin{matrix} -n & -n-\beta \\ & \alpha+1 \end{matrix} \middle| \frac{x-1}{x+1} \right) \\ &= \binom{n+\alpha}{n} \left(\frac{1+x}{2}\right)^n \sum_{k=0}^{n} \frac{(-n)_k(-n-\beta)_k}{(\alpha+1)_k\,k!} \frac{(x-1)^k}{(x+1)^k} \\ &= \binom{n+\alpha}{n} \sum_{k=0}^{n} \frac{(-1)^k(-n)_k(-n-\beta)_k}{(\alpha+1)_k\,k!} \left(\frac{1-x}{2}\right)^k \left(\frac{1+x}{2}\right)^{n-k}. \end{aligned}$$

This is now used to produce a second representation for the Jacobi polynomial.

Lemma 3. The Jacobi polynomial is given by

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k} \left(\frac{x-1}{2}\right)^k \left(\frac{1+x}{2}\right)^{n-k}.$$

Proof. Comparing the desired expression with the last formula before the statement of the Lemma shows that the result is equivalent to prove the identity

$$\binom{n+\alpha}{n} \frac{(-n)_k\,(-n-\beta)_k}{(\alpha+1)_k\,k!} = \binom{n+\alpha}{n-k}\binom{n+\beta}{k}. \tag{12}$$

This follows as before transforming Pochhammer and binomial coefficients in terms of gamma values. □

The next statement is the main result in this section. The notation $[x^n]f(x)$ is used for the coefficient of x^n in the series expansion of f.

Theorem 4. The identity

$$\begin{aligned} [x^n]\,(1-x)^n(1+tx)^\lambda P_n^{(\alpha,\beta)}\left(\frac{1+x}{1-x}\right) &= \frac{(\beta+1)_n}{n!} \sum_{k=0}^{n} \binom{n}{k}\binom{\lambda}{k}\binom{n+\alpha}{k} \frac{k!\,t^k}{(\beta+1)_k} \\ &= \sum_{j=0}^{n} \binom{n+\alpha}{j}\binom{\lambda}{j}\binom{n+\beta}{n-j} t^j, \end{aligned}$$

holds.

Proof. Using the expression for the Jacobi polynomial in Lemma 3 gives

$$(1-x)^n(1+tx)^\lambda P_n^{(\alpha,\beta)}\left(\frac{1+x}{1-x}\right) = (1+tx)^\lambda \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k} x^k. \tag{13}$$

Now

$$(1+tx)^\lambda = \sum_{j=0}^{\lambda} \binom{\lambda}{j} t^j x^j \tag{14}$$

and we need to find the coefficient of x^n in the expansion of the product

$$\left(\sum_{j=0}^{\lambda} \binom{\lambda}{j} t^j x^j \right) \times \left(\sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} x^k \right). \tag{15}$$

This is given by taking the j^{th} term on the left factor with the k^{th} term on the right, subject to $j+k=n$. This produces

$$\sum_{j=0}^{n} \binom{\lambda}{j} \binom{n+\alpha}{j} \binom{n+\beta}{n-j} t^j. \tag{16}$$

This is the second claim. To obtain the first form of the identity, simply write the binomial coefficients in terms of Pochhammer. □

The special case $\lambda = n$, $\alpha = \beta = 0$ and $t = 1$ gives the next result.

Corollary 5. The Franel numbers are given by

$$\sum_{k=0}^{n} \binom{n}{k}^3 = [x^n]\,(1-x^2)^n P_n^{(0,0)} \left(\frac{1+x}{1-x} \right). \tag{17}$$

The polynomial

$$P_n^{(0,0)}(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k}^2 (1-x)^k (1+x)^{n-k} \tag{18}$$

is the `Legendre polynomial`, usually denoted by $P_n(x)$.

References

[1] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products.* Edited by D. Zwillinger and V. Moll. Academic Press, New York, 8th edition, 2015.

[2] M. E. H. Ismail. Sums of products of binomial coefficients. *Ars Combinatoria*, 101:187–192, 2011.