THE INTEGRAL OF A RATIONAL FUNCTION

The question considered here is to produce an explicit form of the integral

$$I_n = \int_0^\infty \frac{dx}{(x^2+1)^n}. \tag{1}$$

Using `Mathematica` one obtains the data

1	2	3	4	5	6	7	8	9	10
$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{3\pi}{16}$	$\frac{5\pi}{32}$	$\frac{35\pi}{256}$	$\frac{63\pi}{512}$	$\frac{231\pi}{2048}$	$\frac{429\pi}{4096}$	$\frac{6435\pi}{65536}$	$\frac{12155\pi}{131072}$

for $1 \leq n \leq 10$. The goal is to produce a recurrence for this integral. But first we illustrate the **peeling method**.

The peeling method consists of using `Mathematica` to obtain data for I_n and use it to **guess** a formula for I_n.

The first few values of I_n contains a factor of π, so it seems a good idea to define

$$W_n = \frac{1}{\pi} I_n. \tag{2}$$

The first few values of W_n are

$$\frac{1}{2} \quad \frac{1}{4} \quad \frac{3}{16} \quad \frac{5}{32} \quad \frac{35}{256} \quad \frac{63}{512} \quad \frac{231}{2048} \quad \frac{429}{4096} \quad \frac{6435}{65536} \quad \frac{12155}{131072}$$

and now we need to identify these rational numbers.

Some information about the denominators is easy to obtain: the list

$$2 \quad 4 \quad 16 \quad 32 \quad 256 \quad 512 \quad 2048 \quad 4096 \quad 65536 \quad 131072$$

show that they all are powers of 2. The corresponding exponents are

$$1 \quad 2 \quad 4 \quad 5 \quad 8 \quad 9 \quad 11 \quad 12 \quad 16 \quad 17$$

and from here it seems that

$$R_n = 2^{2n-1} \times W_n \tag{3}$$

is an integer. The first few values are

$$1 \quad 2 \quad 6 \quad 20 \quad 70 \quad 252 \quad 924 \quad 3432 \quad 12870 \quad 48620$$

A search in OEIS shows that

$$R_n = \binom{2n-2}{n-1}. \tag{4}$$

A new approach to guessing the formula for I_n. A second way to guess the value (4) is explained now: use `Mathematica` to compute the value R_{50}. The answer is

$$R_{50} = 25477612258980856902730428600 \tag{5}$$

that is a 29 digit number. In its factored form, this number is

$$R_{50} = 97 \cdot 89 \cdot 83 \cdot 79 \cdot 73 \cdots 29 \cdot 19 \cdot 17 \cdot 13 \cdot 5^2 \cdot 3^2 \cdot 2^3 \tag{6}$$

and we will use this form to guess what R_{50} should be. The fact that its factorization contains the primes 97, 89, 83, 79, 73 suggests a relation between R_{50} and 100!. Therefore we compute

$$Y_{50} = \frac{R_{50}}{100!}. \tag{7}$$

This turns out to be the reciprocal of an integer, so it is better to compute

$$Z_{50} = \frac{100!}{R_{50}}. \tag{8}$$

This is a 129 digits number and its prime factorization is

$$Z_{50} = 47^2 \cdot 43^2 \cdot 41^2 \cdot 37^2 \cdot 31^2 \cdots 5^{22} \cdot 3^{46} \cdot 2^{94} \tag{9}$$

that is, all primes in the range 51 to 100 have disappeared. Also the exponents of the primes up to 50 are 2. This suggests that Z_{50} is related to $50!^2$. Therefore we compute

$$U_{50} = \frac{Z_{50}}{50!^2} \tag{10}$$

and `Mathematica` gives

$$U_{50} = \frac{99}{25}. \tag{11}$$

To guess a formula for U_n form the table of values for n from 50 tp 54 to obtain the values

50	51	52	53	54
99/25	202/51	103/26	210/53	107/27

The data suggest to define

$$V_n = nU_n \tag{12}$$

that gives

50	51	52	53	54
198	202	206	210	214

that seems to fit the formula

$$V_n = 4n - 2. \tag{13}$$

This is equivalent to (4), as before. Repeating this calculation for other values of n, confirms (4).

Guess. The following formula is true:

$$I_n = \int_0^\infty \frac{dx}{(x^2+1)^n} = \frac{\pi}{2^{2n-1}} \binom{2n-2}{n-1}. \tag{14}$$

The form of the guess shows that it is better to introduce J_n by

$$J_n = \int_0^\infty \frac{dx}{(x^2+1)^{n+1}} \tag{15}$$

and the guess becomes

$$J_n = \frac{\pi}{2^{2n+1}}\binom{2n}{n}. \tag{16}$$

To prove (16) we try to obtain a recurrence: from

$$J_{n+1} = \int_0^\infty \frac{dx}{(x^2+1)^{n+2}} \tag{17}$$

and write the numerator as

$$1 = (x^2+1) - x^2 \tag{18}$$

to produce

$$\begin{aligned} J_{n+1} &= \int_0^\infty \frac{dx}{(x^2+1)^{n+2}} \\ &= \int_0^\infty \frac{(x^2+1)-x^2}{(x^2+1)^{n+2}}\,dx \\ &= \int_0^\infty \frac{dx}{(x^2+1)^{n+1}} - \int_0^\infty \frac{x^2\,dx}{(x^2+1)^{n+2}} \\ &= J_n - \int_0^\infty \frac{x^2\,dx}{(x^2+1)^{n+2}}. \end{aligned} \tag{19}$$

To find a recurrence we need to relate the last integral above in terms of the integral J_n. Define

$$X_n = \int_0^\infty \frac{x^2\,dx}{(x^2+1)^{n+2}}, \tag{20}$$

so that the previous statement reads

$$J_{n+1} = J_n - X_n. \tag{21}$$

Now write

$$\begin{aligned} X_n &= \int_0^\infty \frac{x}{2}\cdot\frac{2x}{(x^2+1)^{n+2}}\,dx \\ &= \int_0^\infty \frac{x}{2}\cdot\frac{d}{dx}\left[\frac{1}{-(n+1)(x^2+1)^{n+1}}\right]dx \end{aligned} \tag{22}$$

Integrating by parts and checking that the boundary terms vanish gives

$$X_n = \frac{1}{2(n+1)}\int_0^\infty \frac{dx}{(x^2+1)^{n+1}}, \tag{23}$$

that is,

$$X_n = \frac{J_n}{2(n+1)}. \tag{24}$$

Replacing in (21) yields

$$J_{n+1} = J_n - \frac{J_n}{2(n+1)} = \frac{2n+1}{2(n+1)}J_n. \tag{25}$$

To prove the identity (16) one could proceed by induction. It becomes easier if we introduce the new unknown Y_n by

$$J_n = Y_n \times \frac{\pi}{2^{2n+1}} \binom{2n}{n}. \tag{26}$$

Substituting in (25) produces the recurrence

$$Y_{n+1} = Y_n. \tag{27}$$

The initial condition $Y_0 = 1$ then implies $Y_n \equiv 1$ for all $n \in \mathbb{N}$. Formula (16) has been established.

Theorem 1. *Let $n \in \mathbb{N}$. Then*

$$J_n = \int_0^\infty \frac{dx}{(x^2+1)^{n+1}} \tag{28}$$

is given by

$$J_n = \frac{\pi}{2^{2n+1}} \binom{2n}{n}. \tag{29}$$