THE EVALUATION OF A TRIGONOMETRIC INTEGRAL

The question considered here is to produce an explicit form of the integrals

$$S_n = \int_0^{\pi/2} \sin^n x \, dx \tag{1}$$

and

$$C_n = \int_0^{\pi/2} \cos^n x \, dx. \tag{2}$$

The fact that

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \tag{3}$$

shows that

$$C_n = S_n. \tag{4}$$

Using `Mathematica` one obtains the data

$$\pi/2 \quad 1 \quad \pi/4 \quad 2/3 \quad 3\pi/16 \quad 8/15 \quad 5\pi/32 \quad 16/35 \quad 35\pi/256$$

for $0 \leq n \leq 8$. This suggests to separate the discussion according to the parity of n. Therefore, define

$$I_n = \int_0^{\pi/2} \sin^{2n} x \, dx = \int_0^{\pi/2} \cos^{2n} x \, dx. \tag{5}$$

The goal is to produce a recurrence for this integral. But first we illustrate the **peeling method**.

Before explaing this method, observe that the change of variables $x = \tan t$ converts (5) to the rational form

$$I_n = \int_0^{\infty} \frac{dx}{(x^2+1)^{n+1}}. \tag{6}$$

The peeling method consists of using `Mathematica` to obtain data for I_n and use it to **guess** a formula for I_n.

The first few values of I_n contains a factor of π, so it seems a good idea to define

$$W_n = \frac{1}{\pi} I_n. \tag{7}$$

The first few values of W_n are

$$1/2 \quad 1/4 \quad 3/16 \quad 5/32 \quad 35/256 \quad 63/512 \quad 231/2048 \quad 429/4096 \quad 6435/65536$$

and now we need to identify these rational numbers.

Some information about the denominators is easy to obtain: the list

$$2 \quad 4 \quad 16 \quad 32 \quad 256 \quad 512 \quad 2048 \quad 4096 \quad 65536$$

show that they all are powers of 2. The corresponding exponents are

$$1 \quad 2 \quad 4 \quad 5 \quad 8 \quad 9 \quad 11 \quad 12 \quad 16$$

and from here it seems that

$$R_n = 2^{2n+1} \times W_n \tag{8}$$

is an integer. The first few values are

1 2 6 20 70 252 924 3432 12870

A search in OEIS shows that

$$R_n = \binom{2n}{n}. \tag{9}$$

A second form of guessing (9) is provided later.

We conclude with the following:

Guess. The following formula is true:

$$I_n = \int_0^{\pi/2} \sin^{2n} x \, dx = \frac{\pi}{2^{2n+1}} \binom{2n}{n}. \tag{10}$$

The next step is to prove this guess. As before, we try to find a recurrence. This will come the basic algebraic relation

$$\sin^2 x + \cos^2 x = 1 \tag{11}$$

and integration by parts.

Write this as

$$\begin{aligned} I_n &= \int_0^{\pi/2} \sin^2 x \times \sin^{2n-2} x \, dx \\ &= \int_0^{\pi/2} (1 - \cos^2 x) \times \sin^{2n-2} x \, dx \\ &= \int_0^{\pi/2} \sin^{2n-2} x \, dx - \int_0^{\pi/2} \cos^2 x \times \sin^{2n-2} x \, dx \\ &= I_{n-1} - \int_0^{\pi/2} \cos^2 x \times \sin^{2n-2} x \, dx. \end{aligned} \tag{12}$$

Call this last integral

$$J = \int_0^{\pi/2} \cos^2 x \times \sin^{2n-2} x \, dx. \tag{13}$$

Now write the integrand as

$$\cos^2 x \times \sin^{2n-2} x = \cos x \times \cos x \sin^{2n-2} x \tag{14}$$

$$= \cos x \times \frac{d}{dx} \left(\frac{1}{2n-1} \sin^{2n-1} x \right) \tag{15}$$

so J becomes

$$J = \int_0^{\pi/2} \cos x \times \frac{d}{dx} \left(\frac{1}{2n-1} \sin^{2n-1} x \right) dx. \tag{16}$$

Integrate by parts and observe that the boundary terms vanish. This gives

$$\begin{aligned} J &= -\int_0^{\pi/2} (-\sin x) \times \left(\frac{1}{2n-1} \sin^{2n-1} x \right) dx \\ &= \frac{1}{2n-1} \int_0^{\pi/2} \sin^{2n} x \, dx \\ &= \frac{1}{2n-1} I_{2n}. \end{aligned}$$

Then (12) becomes

$$I_n = I_{n-1} - \frac{1}{2n-1} I_n \tag{17}$$

that leads to

$$I_n = \frac{2n-1}{2n} I_{n-1}. \tag{18}$$

From here you can prove the value

$$I_n = \frac{\pi}{2^{2n+1}} \binom{2n}{n} \tag{19}$$

using the following nice trick.

Define Y_n by the relation

$$I_n = Y_n \times \frac{\pi}{2^{2n+1}} \binom{2n}{n} \tag{20}$$

and replace in (18) to obtain

$$Y_{n+1} = Y_n. \tag{21}$$

This can be solved to get

$$Y_n \equiv 1. \tag{22}$$

The proof of (10) is complete.

A new approach to guessing the formula for I_n. A second way to guess the value (9) is explained now: use `Mathematica` to compute the value R_{50}. The answer is

$$R_{50} = 1008913445455641933334812497256 \tag{23}$$

that is a 30 digit number. In its factored form, this number is

$$R_{50} = 97 \cdot 89 \cdot 83 \cdot 79 \cdot 73 \cdots 29 \cdot 19 \cdot 17 \cdot 13 \cdot 11 \cdot 3^4 \cdot 2^3 \tag{24}$$

and we will use this form to guess what R_{50} should be. The fact that its factorization contains the primes 97, 89, 83, 79, 73 suggests a relation between R_{50} and 100!. Therefore we compute

$$Y_{50} = \frac{R_{50}}{100!}. \tag{25}$$

This turns out to be the reciprocal of an integer, so it is better to compute

$$Z_{50} = \frac{100!}{R_{50}}. \tag{26}$$

This is a 129 digits number and its prime factorization is

$$Z_{50} = 47^2 \cdot 43^2 \cdot 41^2 \cdot 37^2 \cdot 31^2 \cdots 5^{24} \cdot 3^{44} \cdot 2^{94} \tag{27}$$

that is, all primes in the range 51 to 100 have disappeared. Also the exponents of the primes up to 50 are 2. This suggests that Z_{50} is related to $50!^2$. Therefore we compute

$$U_{50} = \frac{Z_{50}}{50!^2} \tag{28}$$

and `Mathematica` gives

$$U_{50} = 1. \tag{29}$$

This is equivalent to

$$R_{50} = \frac{100!}{50!^2} = \binom{100}{50}. \tag{30}$$

Repeating this calculation for other values of n, also gives

$$U_n = 1 \tag{31}$$

that gives (9).

Theorem 1. *The integral*

$$I_n = \int_0^{\pi/2} \sin^{2n} x \, dx \tag{32}$$

has the value

$$I_n = \frac{\pi}{2^{2n+1}} \binom{2n}{n}. \tag{33}$$