LARRY GLASSER'S THEOREM FOR BEUKERS INTEGRALS

In [1], the author established the identity

$$\int_0^1 \int_0^1 f(xy)\,dx\,dy = -\int_0^1 \ln s\, f(s)\,ds. \tag{1}$$

Taking $f(s) = 1/(1-s)$, this produces the simplest Beukers' integral

$$\int_0^1 \int_0^1 \frac{dx\,dy}{1-xy} = \zeta(2). \tag{2}$$

To prove the formula, observe that by symmetry

$$I = \int_0^1 \int_0^1 f(xy)\,dx\,dy = 2 \int \int_R f(xy)\,dx\,dy \tag{3}$$

where R is the interior of the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$. Make the change of variables

$$u = xy, \quad t = x - y \tag{4}$$

with jacobian

$$J = \left| \det \begin{pmatrix} y & x \\ 1 & -1 \end{pmatrix} \right| = x + y = \frac{1}{\sqrt{t^2+4u}}. \tag{5}$$

The region R is mapped onto the interior of the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$. Therefore

$$I = 2 \int_0^1 \int_0^{1-u} \frac{f(u)}{\sqrt{t^2+4}}\,dt\,du. \tag{6}$$

The change of variables $t = 2\sqrt{u}y$ gives

$$\begin{aligned} \int_0^{1-u} \frac{f(u)}{\sqrt{t^2+4}}\,dt &= \int_0^{(1-u)/2\sqrt{u}} \frac{dy}{\sqrt{y^2+1}} \\ &= \sinh^{-1}\left(\frac{1-u}{2\sqrt{u}}\right). \end{aligned}$$

This implies

$$I = 2 \int_0^1 f(u) \sinh^{-1}\left(\frac{1-u}{2\sqrt{u}}\right) du. \tag{7}$$

In order to transform this integral, we would like to introduce a new variable x such that

$$\frac{1-u}{2\sqrt{u}} = \sinh x. \tag{8}$$

Squaring this gives a quadratic equation for u with solutions

$$\begin{aligned} u &= 1 \pm 2\sinh x \cosh x + 2\sinh^2 x \\ &= 1 \pm \frac{e^{2x} - e^{-2x}}{2} + \frac{e^{2x} - 2 + e^{-2x}}{2} \\ &= \frac{1}{2}\left(\pm(e^{2x} - e^{-2x}) + (e^{2x} + e^{-2x})\right). \end{aligned}$$

Choosing the minus sign gives $u = e^{-2x}$ with x moving from 0 to $+\infty$ (the choice of plus sign gives the same result). This implies

$$I = 4\int_0^\infty x e^{-2x} f(e^{-2x})\, dx. \tag{9}$$

The change of variables $s = e^{-2x}$ gives

$$\int_0^1 \int_0^1 f(xy)\, dx\, dy = -\int_0^1 \ln s\, f(s)\, ds, \tag{10}$$

as claimed.

References

[1] M. L. Glasser. A note on Beukers' and related double integrals. *Amer. Math. Monthly*, 126:361–363, 2019.