THE EXPANSION OF THE DISCRIMINANT

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In class we have talked about *modular forms* of weight k. These are functions $f : \mathbb{H} \to \mathbb{C}$ that are holomorphic in $\mathbb{H}$ and also at $i\infty$ (this means that f has an expansion $f(q) = \sum_{n=0}^{\infty} a_n q^n$, where $q(z) = e^{2\pi i z}$. The *modularity condition* required for f is that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z). \tag{1}$$

for every a, b, c, $d \in \mathbb{Z}$ with $ad - bc = 1$. This is the same as requiring $f(\gamma z) = (cz+d)^k f(z) \quad$ for every $\gamma \in \Gamma$. For now we will require k to be a non-negative even integer. The class of modular functions of weight k is denoted by $\mathfrak{M}_k$.

The *Eisenstein series* defined by

$$E_{2k}(z) = \frac{1}{2\zeta(2k)} \sum \frac{1}{(mz+n)^{2k}} \tag{2}$$

where the sum extends over all integers m, n except the pair $(0,0)$. The series converges to a holomorphic function on $\mathbb{H}$ (your homework) and has the expansion

$$E_{2k}(z) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n. \tag{3}$$

The values of the first Bernoulli numbers are

4	6	8	10	12	14	16	18	20
$-\frac{1}{30}$	$\frac{1}{42}$	$\frac{-1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$	$\frac{43867}{798}$	$-\frac{174611}{330}$

It was shown in class that the space $\mathfrak{M}_0 = \mathbb{C}$, that $\mathfrak{M}_2 = \{0\}$ and that, for $k = 4, 6, 8, 10, 14$, the vector space $\mathfrak{M}_k$ is of dimension 1 and is generated by E_k. Observe that E_4^3 and E_6^2 are both in $\mathfrak{M}_{12}$ and both have the value 1 at $i\infty$. Therefore

$$\Delta(z) = C\left(E_4^3(z) - E_6^2(z)\right) \tag{4}$$

is in $\mathfrak{M}_{12}$ for any $C \in \mathbb{C}$. Also, Δ and E_{12} are linearly independent, since $\Delta(i\infty) = 0$ and $E_{12}(i\infty) = 1$.

Now we look at the expansions at $i\infty$. Start with

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \text{ and } E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n. \tag{5}$$

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Therefore, the expansion of Δ at $i\infty$ is

$$\frac{\Delta(z)}{C} = \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \right)^3 - \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right)^2 . \tag{6}$$

Introduce the notation $u = \sum_{n=1}^{\infty} \sigma_3(n) q^n$ and $v = \sum_{n=1}^{\infty} \sigma_5(n) q^n$. Then $\frac{\Delta(z)}{C} = \left[(1+240u)^3 - (1-504v)^2 \right]$. and expanding gives

$$\frac{\Delta(z)}{C} = 144 \left(5u - 1200u^2 + 96000u^3 + 7v + 1764v^2 \right). \tag{7}$$

The coefficient of q in the expansion of $\Delta(z)/C$ comes from the term $5u+7v$, since the remaining terms start with q^2. In the series for $5u+7v$, this coefficient is $5\sigma_3(1) + 7\sigma_5(1) = 12$. The constant C is chosen to normalize the expansion of Δ to start with $1 \cdot q$. This is done by choosing $C = 1/1728$.

Definition. The function Δ is defined by $\Delta(z) = \frac{1}{1728} \left(E_4^3(z) - E_6^2(z) \right)$. This is called the **discriminant** (for reasons that will become clear soon). The coefficients in the expansion

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n, \tag{8}$$

are called the *Ramanujan coefficients*.

The equation (7) is now written as

$$\Delta(z) = \frac{1}{12} \left(5u - 1200u^2 + 96000u^3 + 7v + 1764v^2 \right) \tag{9}$$

$$= \frac{1}{12}(5u+7v) - 100u^2 + 8000u^3 + 147v^2. \tag{10}$$

Theorem. The coefficients $\tau(n)$ are integers.

Proof. The series u^2, u^3 and v^2 have integer coefficients. Therefore, the result follows from the fact that $5u+7v$ has coefficients that are multiples of 12. The result now follows from the congruence

$$5\sigma_3(n) + 7\sigma_5(n) \equiv 0 \bmod 12. \tag{11}$$

This can be written as $5\sigma_3(n) \equiv 5\sigma_5(n) \bmod 12$. Cancelling the 5, the result now follows from the congruence $\sigma_3(n) \equiv \sigma_5(n) \bmod 12$. This is left to you as an exercise. It comes down to prove that $d^3(d-1)(d+1)$ is a multiple of 12. □

The Ramanujan coefficients satisfy many identities that will be discussed in class. You will see soon that $\tau(n)$ is a multiplicative function and satisfies the estimate $|\tau(p)| \leq 2p^6$, for p prime. The improvement $|\tau(p)| \leq 2p^{11/2}$, was proved by P. Deligne in 1974 as a corollary of his proof of the so-called *Weil conjectures*. This is quite hard.

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