THE SUM OF THE POWERS OF INTEGERS

The first question of interest is to obtain a *closed-form* expression for the sums

$$S_a(n) := \sum_{k=1}^{n} k^a. \tag{1}$$

A first approach to a definition of closed-form is this: what we want is a function $F(x)$, that depends on the parameter a such that

$$F(n) = S_a(n) \text{ for all } n \in \mathbb{N}. \tag{2}$$

This is easy to do for $a = 1$. In this case the sum is

$$S_1(n) := \sum_{k=1}^{n} k \tag{3}$$

and we all have seen the trick of summing in reverse:

$$\begin{aligned} S_1(n) &= 1 + 2 + \cdots + n \\ S_1(n) &= n + (n-1) + \cdots + 1. \end{aligned} \tag{4}$$

Now do a *vertical sum* to see that

$$2S_1(n) = (n+1) + (n+1) + \cdots + (n+1) = n(n+1) \tag{5}$$

and this gives

$$S_1(n) = \frac{n(n+1)}{2}. \tag{6}$$

The question is how to guess an expression for the function $S_a(n)$.

Start with the case $a = 2$ and consider the sum

$$S_2(n) = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2. \tag{7}$$

The first few values are given in the next table:

n	1	2	3	4	5	6	7	8	9	10
$S_2(n)$	1	5	14	30	55	91	140	204	285	385

The question is what kind of function can one fit to this data. It is natural to start with the simplest kind of functions: *polynomial*.

J. L. Lagrange figured out how to obtain a formula for a polynomial $y = P(x)$ with the property

$$P(x_i) = y_i \tag{8}$$

for a given set of m points $\{(x_i, y_i) : 1 \leq i \leq m\}$.

Start with a simpler goal: fix i in the range $1 \leq i \leq m$ and then find a polynomial $P_i(x)$ such that

$$P_i(x) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{if } x = x_j \text{ with } j \neq i. \end{cases} \tag{9}$$

This is easy to achieve: the fact that $P_i(x)$ has zeros at x_j, $j \neq i$, suggests that polynomial

$$A(x) = \prod_{j \neq i} (x - x_j) \tag{10}$$

and the polynomial

$$P_i(x) = \frac{A(x)}{A(x_i)} \tag{11}$$

satisfies (9). Observe that the degree of $P_i(x)$ is one less than the numbers of points; that is

$$\text{deg} P_i = m - 1. \tag{12}$$

The solution to (8) is now obtained by linearity:

$$P(x) = \sum_{r=1}^{m} y_r P_r(x) \tag{13}$$

satisfies

$$P(x_i) = \sum_{r=1}^{m} y_r P_r(x_i) = y_i. \tag{14}$$

The discussion is summarized in the next theorem.

Theorem 1. *Let* $\{(x_i, y_i) : 1 \leq i \leq m\}$ *be a collection of* m *data points, with* $x_i \neq x_j$ *for* $i \neq j$. *Define*

$$A_i(x) = \prod_{j \neq i} (x - x_j) \quad \text{for } 1 \leq i \leq m. \tag{15}$$

Then

$$L_m(x) = \sum_{r=1}^{m} y_r \frac{A_r(x)}{A_r(x_r)} \tag{16}$$

interpolates the data. This means

$$L_m(x_i) = y_i \quad \text{for } 1 \leq i \leq m. \tag{17}$$

Now use this formula to interpolate the data generated before for the sum $S_2(n)$. Clearly

$$L_1(x) = 1 \tag{18}$$

interpolates the first pair of points:

$$L_1(1) = 1. \tag{19}$$

Now for the first two pairs

$$\{(1, 1),\ (2, 5)\} \tag{20}$$

the interpolating polynomail is

$$L_2(x) = 1\frac{x-2}{1-2} + 5\frac{x-1}{2-1} = 4x - 3 \tag{21}$$

and for the first three data points

$$\begin{aligned} L_3(x) &= 1 \cdot \frac{(x-2)(x-3)}{(1-2)(1-3)} + 5 \cdot \frac{(x-1)(x-3)}{(2-1)(2-3)} + 14 \cdot \frac{(x-1)(x-2)}{(3-1)(3-2)} \\ &= \frac{5}{2}x^2 - \frac{7}{2}x + 2. \end{aligned} \tag{22}$$

The formulas for the polynomial $L_m(x)$ can be computed in `Mathematica` in the following form: define the function S_2 by

$$S[n\_] := Sum[k^2, \{k, 1, n\}] \tag{23}$$

and generate the table of data by

$$data[m\_] := Table[\{i, S[i]\}, \{i, 1, m\}] \tag{24}$$

then compute the interpolating polynomial by

$$L[m\_, n\_] := InterpolatingPolynomial[data[m], n] \tag{25}$$

(the answer is given as a function of the variable n) and to get in expanded form use

$$Expand[L[m, n]] \tag{26}$$

The following list gives the values of the interpolating polynomial as a function of the number m of data points employed. The variable is written as n.

m	$L_m(n)$
1	1
2	$4n-3$
3	$\frac{5}{2}n^2 - \frac{7}{2}n + 2$
4	$\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$
5	$\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$
6	$\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$
7	$\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$

The observation from this table is that the degree of the interpolating polynomial does not grow with the number of data points (this should be surprising) and that its form does not change. This leads to the conjecture

$$S_2(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{n(n+1)(2n+1)}{6}. \tag{27}$$

The question now is how does one prove this. Of course, one wants not only to prove this but to learn something from the proof.

Theorem 2. *The sum of the first n squares of the integers is given by*

$$S_2(n) = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \tag{28}$$

This can be proved by induction in an easy manner:

Proof by induction. The result is valid for $n = 1$ since

$$S_2(1) = 1 \tag{29}$$

and

$$\frac{1(1+1)(2\cdot 1+1)}{6} = 1. \tag{30}$$

Now assume (28) and observe that

$$\begin{aligned} S_2(n+1) &= \sum_{k=1}^{n+1} k^2 \\ &= \sum_{k=1}^{n} k^2 + (n+1)^2 \\ &= S_2(n) + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$

and this completes the inductive step. The important point to realize is that the key idea is the recurrence

$$S_2(n+1) = S_2(n) + (n+1)^2. \tag{31}$$

A small variation. Sometimes it becomes easier to prove a statement by induction if it is converted to the form

$$\text{Something} = 1. \tag{32}$$

For example, to prove (28) define

$$T_2(n) = S_2(n) \text{ divided by } \frac{n(n+1)(2n+1)}{6} \tag{33}$$

that is,

$$S_2(n) = T_2(n) \cdot \frac{n(n+1)(2n+1)}{6}. \tag{34}$$

Replacing in (31) becomes

$$T_2(n+1) \cdot \frac{(n+1)(n+2)(2n+3)}{6} = T_2(n) \cdot \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \tag{35}$$

that simplifies to

$$T_2(n+1) = \frac{(2n^2+n)T_2(n) + 6n + 6}{2n^2 + 7n + 6}. \tag{36}$$

Now it becomes clear that if $T_2(n) = 1$, then $T(n+1) = 1$. The proof is complete.

The formulas

$$S_1(n) = \frac{n(n+1)}{2} \text{ and } S_2(n) = \frac{n(n+1)(2n+1)}{6} \tag{37}$$

makes one think that $S_a(n)$ is a polynomial in n of degree $a+1$. A s simple `Mathematica` calculation shows that this is true:

$$\begin{aligned} S_1(n) &= \frac{1}{2}n^2 + \frac{1}{2}n \\ S_2(n) &= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \\ S_3(n) &= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \\ S_4(n) &= \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n \\ S_5(n) &= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 \\ S_6(n) &= \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n. \end{aligned} \tag{38}$$

From this data it becomes clear that $S_a(n)$ should be a polynomial in n, of degree $a+1$ and that the coefficients are rational numbers. The question is how does one prove this.

One of the running themes is that the existence of recurrences is a useful tool to establish nice results. We have already seen this: the sum $S_2(n)$ of the squares satisfies

$$S_2(n+1) = S_2(n) + (n+1)^2. \tag{39}$$

In order to create a recurrence for the sum

$$S_a(n) = \sum_{k=1}^{n} k^a \tag{40}$$

we try to obtain some recurrence for the *summand* k^a. The simplest possible case is to compare $(k+1)^a$ with k^a. Consider first the example $a=2$ for simplicity:

$$(k+1)^2 - k^2 = 2k+1 \tag{41}$$

and adding from $k=1$ to $k=n$, it is observed that the left-hand side *telescopes*, that is,

$$\sum_{k=1}^{n} \left[(k+1)^2 - k^2 \right] = (n+1)^2 - 1. \tag{42}$$

Therefore (41) produces

$$\sum_{k=1}^{n} \left[(k+1)^2 - k^2 \right] = \sum_{k=1}^{n} (2k+1) \tag{43}$$

that gives

$$(n+1)^2 - 1 = 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1, \tag{44}$$

that is written as

$$(n+1)^2 - 1 = 2S_1(n) + S_0(n). \tag{45}$$

Of course $S_0(n) = n$, so we conclude that

$$S_1(n) = \frac{1}{2}n^2 + \frac{1}{2}n, \tag{46}$$

as before.

In the general situation, start with

$$(k+1)^{a+1} - k^{a+1} = \sum_{r=0}^{a} \binom{a+1}{r} k^r \tag{47}$$

where we have used the power $a+1$ because the special case $a = 1$ came from looking at $(k+1)^2 - k^2$. Now sum from $k = 1$ to $k = n$ to produce

$$(n+1)^{a+1} - 1 = \sum_{r=0}^{a} \binom{a+1}{r} \sum_{k=1}^{n} k^r. \tag{48}$$

Isolating the term with $r = a$ gives

$$(n+1)^{a+1} - 1 = \sum_{r=0}^{a-1} \binom{a+1}{r} \sum_{k=1}^{n} k^r + \binom{a+1}{a} \sum_{k=1}^{n} k^a, \tag{49}$$

that produces

$$(a+1)S_a(n) = (n+1)^{a+1} - 1 - \sum_{r=0}^{a-1} \binom{a+1}{r} S_r(n). \tag{50}$$

Now assume (by induction) that $S_r(n)$ is a polynomial in n, of degree $r+1$ and rational coefficients. Then the sum on the right of (50) is a polynomial of degree a. The term $(n+1)^{a+1}$ is of degree $a+1$. Therefore, the right-hand side is a polynomial in n of degree $a+1$. It also clear from here that the leading coefficient is $1/(a+1)$.

Theorem 3. *The sum*

$$S_a(n) := \sum_{k=1}^{n} k^a \tag{51}$$

is a polynomial in the variable n, with rational coefficients, of degree $a+1$ and leading coefficient $1/(a+1)$. These sums satisfy the recurrence

$$S_a(n) = \frac{(n+1)^{a+1} - 1}{a+1} - \frac{1}{a+1} \sum_{r=0}^{a-1} \binom{a+1}{r} S_r(n). \tag{52}$$

It would be nice to have a closed-form of the polynomials $S_a(n)$. In order to find some information about them, consider the list of the coefficients of the first power of n. This begins with

$$\left\{ 1,\ \frac{1}{2},\ \frac{1}{6},\ 0,\ -\frac{1}{30},\ 0,\ \frac{1}{42},\ 0,\ -\frac{1}{30},\ 0,\ \frac{5}{66} \right\} \tag{53}$$

and it seems that every odd term, except the first one, vanishes. The list of the linear coefficients for *even order* sums $S_a(n)$ is

$$\left\{1, \frac{1}{6}, -\frac{1}{30}, \frac{1}{42}, -\frac{1}{30}, \frac{5}{66}, -\frac{691}{2730}, \frac{7}{6}, -\frac{3617}{510}\right\}. \tag{54}$$

There is a great website that allows you to find information about integers. It was created by N. Sloane and it can be found at

`https://oeis.org`

The coefficients in (54) are not integers, but we can enter the list of denominators

$$\{1, 6, 30, 42, 30, 66, 2730, 6, 510, 798, 330\} \tag{55}$$

and we immediatly find that this list agrees with the denominators of the even indexed **Bernoulli numbers** B_{2n}. This is entry $A002445$. The site gives the formula for the *exponential generating function*

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \tag{56}$$

At the moment (unless you continue searching on the web) it is unclear what these number have to do with the original problem, but they look interesting, so we try to find something about them. *This will appear in a different document.*